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S-Duality in N=4 Yang-Mills Theories with General Gauge Groups

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ABSTRACT

’t Hooft construction of free energy, electric and magnetic fluxes, and of the partition function with twisted boundary conditions, is extended to the case of $N = 4$ supersymmetric Yang-Mills theories based on arbitrary compact, simple Lie groups.

The transformation of the fluxes and the free energy under S-duality is presented. We consider the partition function of $N = 4$ for a particular choice of boundary conditions, and compute exactly its leading infrared divergence. We verify that this partition function obeys the transformation laws required by S-duality. This provides independent evidence in favor of S-duality in $N = 4$ theories.

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1 Introduction

Classical Maxwell's equations can be made symmetric in the exchange of the electric and magnetic fields by introducing a magnetic current, together with the electric one, i.e., by introducing magnetic monopoles besides electrically charged particles. In any quantum theory the magnetic charge of the monopoles, g_m, has to obey the Dirac quantization condition, $g_m = 4\pi n/g_e$, where n is an integer and g_e is the electric charge. In Abelian gauge theories the monopoles correspond to singular gauge configuration.

On the other hand, there exist non-Abelian theories where monopoles exist as classical, non-singular configurations of finite energy; an example is the $SO(3)$ gauge theory, with a scalar field in the adjoint representation, breaking the symmetry to $U(1)$. The existence of classical finite-energy monopoles means that there exist magnetically charged particles in the quantum theory. Their mass spectrum is in general difficult to compute since its classical value is modified by perturbative corrections, still, it was conjectured that a theory with monopoles could be symmetric under the exchange of electrically charged particles with monopoles, together with $g_e \leftrightarrow g_m$ [1].

A remarkable simplification happens in the $N = 4$ supersymmetric Yang-Mills theory. There, the supersymmetry algebra [2] gives an exact result for the mass of electrically and/or magnetically charged particles belonging to "short multiplets" (supersymmetry multiplets containing only states of spin not greater than one). It turns out that if a short multiplet has p units of electric charge, and q units of magnetic charge, its square mass is equal to a universal constant times $p^2 g_e^2 + q^2 g_m^2$. The existence of magnetic monopoles, with $p = 0$, $q = 1$, besides the electrically charged ones, and the fact that their mass is symmetric under the exchange $g_m \leftrightarrow g_e$, $p \leftrightarrow q$ gave rise to the hypothesis that $N = 4$ super Yang-Mills is the most likely theory to verify the Montonen-Olive conjecture [2, 3].

This conjecture can be generalized to arbitrary gauge groups following ref. [4]: in general, the duality $g \to 4\pi/g$ transforms the gauge group G into another group (the magnetic group) $\hat{G}$, whose weight lattice is dual to the weight lattice of G.

Arguments in favor of electric-magnetic duality for the S-matrix of $N = 4$ where given in [5], thereby making it plausible that $g \to 4\pi/g$ is a true property of the theory and not only of its mass spectrum. It was also realized in lattice calculations [6] that in the presence of a theta term, electric-magnetic duality and theta shifts, $\theta \to \theta + 2\pi$, could be combined into an $SL(2, Z)$ group acting on the complex parameter

$$S = \frac{4\pi}{g^2} + i\frac{\theta}{2\pi} \tag{1.1}$$

by fractional transformations

$$iS \to \frac{aiS + b}{ciS + d}, \quad a, b, c, d \text{ integers}, \; ad - bc = 1. \tag{1.2}$$

This enlarged symmetry, called S-duality, maps states with, say, $p = 1$, $q = 0$ (elementary states, electrically charged), into states with any p and q relatively prime. Recently, Sen [7] has

constructed the previously unknown states with $p = 2$, $q = 1$; this result gives important new evidence for the S-duality conjecture.

S-duality includes the original strong-weak coupling duality $g \leftrightarrow 4\pi/g$ at zero theta angle. In [8] it was shown that this duality can be used to compute exactly the low-energy effective action of $N = 2$ asymptotically free theories, even though $N = 2$ is not explicitly S-dual. These arguments have been extended to phenomenologically interesting $N = 1$ models [9] and provide a new way of understanding the physics of gauge theories at strong coupling.

It is possible that S-duality is also a symmetry of strings theory [10]. Evidence for this conjecture has been given, by different methods, in [11, 12, 13]. If true, this would be extremely important since S-duality is fundamentally non-perturbative (it relates strong and weak coupling) and does not hold order by order in the string loop expansion; thus, it may give a way of studying the non-perturbative dynamics of strings. Together with T-duality (see [14] for a review) it may also be a part of a much larger (infinite dimensional) stringy symmetry [15].

Moreover, if S-duality is a fundamental symmetry of string theory, it will explain its appearance in gauge theories. That is because the latter can be derived as the $\alpha' \to 0$ limit of string theory (α' is the inverse string tension), and S-duality should hold order by order in the α' expansion.

Most of the evidence for S-duality is classical or semiclassical in nature, and its validity for the full quantum theory relies on the existence of non-renormalization theorems. For this reason, strong-coupling tests of the conjecture are difficult. A strong-coupling test of S-duality was given in [16], where a topological, twisted version of $N = 4$ was defined, and the corresponding partition function computed on various manifolds was found to be S-dual.

In [17] we proposed an ansatz for the thermodynamical free energy of $N = 4$, $SU(2)$ Yang-Mills theory in a finite-size box, and analyzed its transformation properties under S-duality. This free energy depends on the non-Abelian generalization of the electric and magnetic fluxes, defined long ago by 't Hooft [18]. In [17] we computed the free energy in the weak coupling limit, and then proposed a possible extension to any S by requiring S-duality.

In this paper, we define an appropriate partition function (and the corresponding free energy) for $N = 4$ based on an *arbitrary* compact, simple Lie group G (the generalization to any compact group is obvious). Moreover, we *compute* the leading infrared divergence term of the partition function, by a path-integral formulation. This free-energy turns out to be identical with the one proposed in [17], for the case $G = SU(2)$, and turns out to be S-dual for any gauge group. Namely, the partition function transforms as demanded by S-duality.

The electric and magnetic fluxes transform appropriately into other electric and magnetic fluxes for any element of the S-duality group, $SL(2, Z)$. However, when G is non-simply laced, some elements of $SL(2, Z)$ transform "legal" fluxes into "illegal" fluxes. For those believing that $SL(2, Z)$ S-duality originates as a fundamental symmetry of string theory, these might be "good news." This is because non-simply laced gauge groups are impossible in the $D = 4$, $N = 4$ heterotic string compactifications (whose $\alpha' \to 0$ limit gives rise to $N = 4$ Yang-Mills theories in $D = 4$).

The paper is organized as follows. In Section 2, we review the construction of $N = 4$

super Yang-Mills with θ term. Section 3 extends 't Hooft's construction of the free energy of a flux to any compact gauge group (the original construction was given only for $SU(N)$). The new feature that appears for non-simply laced groups is that the gauge algebrae of G and $\hat{G}$ are different, and get interchanged under electric-magnetic duality. Section 4 contains a path-integral computation of the leading infrared divergence of the partition function and of the corresponding free energies. This can be carried out exactly with an appropriate choice of the boundary conditions and gauge fixing, and results in an explicit formula which can be used to test S-duality. In Section 5, we show that the free energy defined in the previous Section obeys factorization, 't Hooft duality, and properly accounts for the shift in electric charge that occurs when the theta angle is changed (Witten's phenomenon [19]). Section 6 contains the explicit transformation laws that the free energies of *any* S-dual theory must obey. These transformation laws arise independently of any particular realization of the free energy, and only depend on the assumption of S-duality symmetry. Still in Section 6, we verify that the free energies obtained in Section 4 do transform as prescribed by S-duality. Section 7 contains a summary and some final remarks. In Appendix A, we set up the notation needed to carry on the construction of the free energy for arbitrary simple Lie groups. Appendix B contains an alternative computation of the partition functions, performed in the Hamiltonian formalism. The computation of its leading infrared divergence can be carried out completely at weak coupling, and it gives rise to the same leading infrared divergence of the path integral computation, thus providing another check of our formulae.

2 $N = 4$ Supersymmetric Yang-Mills Theories

An $N = 4$ super Yang-Mills theory is completely determined by the gauge group, G. The fields in the Lagrangian (L) form a supermultiplet:

$$\begin{gathered}\Phi = (A^a_\mu,\ \ \lambda^a_I,\ \ \phi^a_{IJ}) \\ \mu = 1,2,3,4, \qquad I,J = 1,2,3,4, \qquad a = 1,...,d, \quad d = \dim G.\end{gathered} \tag{2.1}$$

All the fields in (2.1) are in the adjoint representation of the gauge group. The supermultiplet contains a gauge field (spin-1), A^a_μ (μ is a space-time vector index and a is a group index of the adjoint representation), four Weyl spinors (spin-1/2), λ^a_I (I is the so-called "extension index," in the 4 of $SU(4)$, representing the four supersymmetry charges), and six scalars (spin-0), ϕ^a_{IJ}, which obey the condition: $2\phi^a_{IJ} = \epsilon_{IJKL}(\phi^a_{KL})^*$. The Lagrangian, which can be obtained most easily by dimensional reduction of the $N = 1$ Yang-Mills theory in 10 dimensions [20], takes the form

$$\begin{aligned} L &= \frac{1}{4\pi}\mathrm{Re}\, S\Big[\frac{1}{2}F^a_{\mu\nu}F^{a\,\mu\nu} + \bar{\lambda}^{a\,I}\not{D}\lambda^a_I + D_\mu\phi^{a\,IJ}D^\mu\phi^a_{IJ} \\ &\quad + f_{abc}\bar{\lambda}^{a\,I}\phi^b_{IJ}\lambda^{c\,J} + f_{abc}f_{ade}\phi^b_{IJ}\phi^{c\,JK}\phi^d_{KL}\phi^{e\,LI}\Big] \\ &\quad - i\frac{1}{8\pi}\mathrm{Im}\, S F^a_{\mu\nu}\tilde{F}^{a\,\mu\nu}. \end{aligned} \tag{2.2}$$

Here $\phi^{a\,IJ} \equiv (\phi^a_{IJ})^*$, $2\tilde{F}^{a\,\mu\nu} = \epsilon^{\mu\nu\sigma\rho}F^a_{\sigma\rho}$, and f_{abc} are the structure constants of G. The Cartan-Killing metric is δ_{ab}. By eq. (2.2), one finds the relation between S, the coupling constant g, and theta angle θ:

$$S = \alpha^{-1} + ia, \qquad \alpha = \frac{g^2}{4\pi}, \qquad a = \frac{\theta}{2\pi}. \tag{2.3}$$

The first three terms in L are the kinetic terms of the gauge fields, spinors and scalars, respectively, with covariant derivatives due to gauge symmetry. The fourth term in L is the Yukawa interaction, and the fifth term is the scalar potential. The latter has flat directions when the scalar fields get Vacum Expectation Values (VEVs) in the Cartan Sub-Algebra (CSA). In $N = 4$ super Yang-Mills theory, the potential does not receive perturbative quantum corrections [21, 22]. Indeed, $N = 4$ is so rigid as to forbid even non-perturbative corrections [22]. This result agrees with explicit instanton calculations, as discussed in the second paper of ref. [22]. The last term in L is a topological invariant.

Our aim is to find appropriate gauge invariant quantities which are simple enough to be calculable, yet non-trivial, i.e. carrying some dynamical information about the theory. One possibility is to follow the strategy outlined by 't Hooft in [18], where the non-Abelian equivalent of the electric and magnetic fluxes are defined. Ref. [18] also contains the definition of the free energies of electric and magnetic fluxes, in terms of appropriate functional integrals with twisted boundary conditions, when the gauge theory contains only fields in the adjoint of the gauge group, and the group itself is $SU(N)$. The restriction to the adjoint representation is automatic in $N = 4$, since all matter fields belong to the same multiplet and, therefore, their representation is the same as that of the gauge vectors. On the other hand, we would like to generalize 't Hooft's prescription to arbitrary gauge groups.

Before doing this in the next Section, let us recall that 't Hooft free energies may be used to classify the phases of pure Yang-Mills. In our case the presence of scalars, and of flat directions in the scalar potential, alters significantly the picture. Indeed, by choosing for instance periodic boundary conditions for the scalars one ends up evaluating the free energy in a mixed phase. Thus 't Hooft boxes may not be enough to distinguish different dynamical realizations of the gauge symmetry.

3 't Hooft Box with Twisted Boundary Conditions, the Free Energy, and the S-Duality Conjecture

The notations and normalizations we shall use in this work are introduced in Appendix A. The reader interested in the algebraic details might find it useful to read Appendix A at this point.

We now want to evaluate the Euclidean functional integral in a box of sides $a_\mu = (a_1, a_2, a_3, a_4)$ and with twisted boundary conditions

$$W[\hat{\mathbf{k}}, \hat{\mathbf{m}}] = \int [dA^a_\mu \, d\lambda^a_I \, d\phi^a_{IJ}] \exp(-\int d^4x L), \tag{3.1}$$

where the Lagrangian L is given in (2.2), and

$$\hat{\mathbf{k}} \equiv (\hat{k}_1, \hat{k}_2, \hat{k}_3), \qquad \hat{\mathbf{m}} \equiv (\hat{m}_1, \hat{m}_2, \hat{m}_3),$$
$$\hat{k}_i, \hat{m}_j \in \frac{\hat{\Lambda}_W}{\hat{\Lambda}_R} \simeq C. \tag{3.2}$$

Here $i, j = 1, 2, 3$ label the spatial coordinates, and $\hat{k}_i, \hat{m}_j$ are vectors in the dual weight lattice modulo the root lattice, for any i and j, namely, each represents an element of the center C (see Appendix A, in particular, eqs. (A.16),(A.21) and (A.29)). The center elements $\hat{k}_i, \hat{m}_j$ are defined through the boundary conditions of the fields, which read

$$\Phi(x + a_\mu e_\mu) = (-)^F \Omega_\mu(x)\Phi(x), \tag{3.3}$$

where e_μ is a unit vector in the μ direction, and repeated indices are not summed. Φ and $\Omega\Phi$ denote generically a field of the supermultiplet (2.1) and its gauge transform under Ω, respectively; F is the fermion number: $F = 0$ on bosons and $F = 1$ on fermions. In other words, the boundary conditions for all bosonic fields are periodic, up to a gauge transformation, whereas fermions are *antiperiodic*, up to a gauge transformation. Going from x to $x + a_\nu e_\nu + a_\mu e_\mu$, $\mu \neq \nu$, in two different ways – either in the ν direction first and then in the μ direction or vice-versa – implies the consistency conditions:

$$\begin{aligned} \Omega_\mu(x + a_\nu e_\nu)\Omega_\nu(x) &= \Omega_\nu(x + a_\mu e_\mu)\Omega_\mu(x) z_{\mu\nu}, \\ z_{\mu\nu} &\in C. \end{aligned} \tag{3.4}$$

The relation (3.4) is consistent, because a constant gauge transformation in the center acts trivially on fields in the adjoint representation. The constant center elements $z_{\mu\nu}$ can be written explicitly as

$$z_{\mu\nu} \equiv z_{\hat{w}_{\mu\nu}} = e^{2\pi i \hat{w}_{\mu\nu}\cdot T}, \qquad \hat{w}_{\mu\nu} \in \frac{\hat{\Lambda}_W}{\hat{\Lambda}_R}. \tag{3.5}$$

Equation (3.4) implies that

$$\Omega_\nu(a_\mu)\Omega_\mu(0) = \Omega_\mu(a_\nu)\Omega_\nu(0) z_{\mu\nu}^{-1} = \Omega_\mu(a_\nu)\Omega_\nu(0) z_{\nu\mu}, \tag{3.6}$$

and from (3.6) and (3.5) we learn that

$$z_{\nu\mu} = z_{\mu\nu}^{-1} \;\Leftrightarrow\; \hat{w}_{\nu\mu} = -\hat{w}_{\mu\nu}. \tag{3.7}$$

Thus, $\hat{w}_{\mu\nu}$ is antisymmetric in the space-time indices.

The elements $\hat{m}_i$ in $W[\hat{\mathbf{k}}, \hat{\mathbf{m}}]$ (3.1) are defined by the twists in the spatial directions:

$$\hat{m}_i \equiv \frac{1}{2}\epsilon_{ijk}\hat{w}_{jk}, \qquad i, j, k = 1, 2, 3. \tag{3.8}$$

$\hat{m}_i$ are interpreted as non-Abelian “magnetic fluxes” [18]. The elements $\hat{k}_i$ in $W[\hat{\mathbf{k}}, \hat{\mathbf{m}}]$ are defined by the twists in the time and space directions:

$$\hat{k}_i \equiv \hat{w}_{4i}, \qquad i = 1, 2, 3. \tag{3.9}$$

$\hat{k}_i$ are interpreted as the dual "electric fluxes." The non-Abelian electric fluxes, $e_i \in \Lambda_W/\Lambda_R$, are linked to $\hat{k}_i$ by the equation [18]:

$$e^{-\beta F[\mathbf{e},\hat{\mathbf{m}}]} = \frac{1}{N^3} \sum_{\hat{\mathbf{k}} \in (\hat{\Lambda}_W/\hat{\Lambda}_R)^3} e^{2\pi i \mathbf{e}\cdot\hat{\mathbf{k}}} W[\hat{\mathbf{k}}, \hat{\mathbf{m}}]. \tag{3.10}$$

Here $\beta \equiv a_4$ is the inverse temperature, $F[\mathbf{e}, \hat{\mathbf{m}}]$ is the free energy of a configuration with electric flux $\mathbf{e}$ and magnetic flux $\hat{\mathbf{m}}$:

$$\mathbf{e} = (e_1, e_2, e_3), \;\; e_i \in \Lambda_W/\Lambda_R, \qquad \hat{\mathbf{m}} = (\hat{m}_1, \hat{m}_2, \hat{m}_3), \;\; \hat{m}_i \in \hat{\Lambda}_W/\hat{\Lambda}_R, \qquad \mathbf{e}\cdot\hat{\mathbf{k}} \equiv \sum_{i=1}^{3} e_i \cdot \hat{k}_i, \tag{3.11}$$

and

$$N = Order(C \simeq \hat{\Lambda}_W/\hat{\Lambda}_R). \tag{3.12}$$

(Recall that the detailed setting of the notations and normalizations is given in Appendix A.) Equation (3.10) means that the free energy is given by the discrete Fourier transform of the functional integrals $W[\hat{\mathbf{k}}, \hat{\mathbf{m}}]$.

The S-duality conjecture is that under the inversion of the complexified coupling constant S in (2.3), the free energies transform as:

$$F[\mathbf{e}, \hat{\mathbf{m}}, 1/S, \mathcal{G}] = F[\hat{\mathbf{m}}, -\mathbf{e}, S, \hat{\mathcal{G}}], \tag{3.13}$$

while under a theta shift of S, they transform as:

$$F[\mathbf{e}, \hat{\mathbf{m}}, S + i, \mathcal{G}] = F[\mathbf{e} + \hat{\mathbf{m}}, \hat{\mathbf{m}}, S, \mathcal{G}]. \tag{3.14}$$

The transformations $S \to 1/S$ and $S \to S + i$ generate the S-duality group, isomorphic to $SL(2, Z)$. We should remark that these transformation laws are expected to hold in any S-duality symmetric theory, independently of the specific form of the partition function. We will discuss S-duality in $N = 4$ Yang-Mills theories in detail in Section 6. But first, in the next Section, we will isolate the leading infrared-divergent contribution to $W[\hat{\mathbf{k}}, \hat{\mathbf{m}}]$ in (3.1), and evaluate the corresponding free energy.

4 Computing the Free Energy

4.1 Computing the Twisted Functional Integral

The twisted functional integral (3.1) is in general too difficult to compute. Moreover, it is plagued by infrared divergences, even in a finite box. They are due to the integration of the scalar zero-modes over the flat directions of the $N = 4$ potential[5]. Fortunately, these

[5] Our realization of eq. (3.4), given in eq. (4.12), implies that the boundary conditions of the scalar fields belonging to the Cartan subalgebra are strictly periodic. This means that one must integrate over all their zero modes.

two difficulties cure each other, in the sense that the most divergent term in eq. (3.1) can be computed exactly, up to a flux-independent constant. This proceeds as follows: in a box of 3-d volume V and at temperature β, we introduce an appropriate infrared regulator $M\beta V$, [6] independent of the coupling constant and theta angle, which cuts off the integral over the scalar zero modes. Then one finds an expansion in M for the partition function

$$W[\hat{\mathbf{k}}, \hat{\mathbf{m}}] = M^n\{w[\hat{\mathbf{k}}, \hat{\mathbf{m}}] + O(M^{-\epsilon})\}, \quad \epsilon > 0, \;\; n > 0. \tag{4.1}$$

If the conjectured S-duality holds for $W[\hat{\mathbf{k}}, \hat{\mathbf{m}}]$, it should also hold order by order in the M expansion and, in particular, for the coefficient of its leading infrared divergence, $w[\hat{\mathbf{k}}, \hat{\mathbf{m}}]$. Thus we find a new test of S-duality. As we shall see below, $w[\hat{\mathbf{k}}, \hat{\mathbf{m}}]$, unlike the full functional integral, can be computed exactly, and its transformation properties under S-duality easily determined.

First of all we must re-examine the standard gauge-fixing procedure and adapt it to the case of twisted boundary conditions. The functional integral and boundary conditions are given in eqs. (3.1), (3.3), (3.4). The functional integral in a given twist sector $\hat{\mathbf{k}}, \hat{\mathbf{m}}$ is extended to all fields $\Phi(x)$ obeying the boundary conditions (3.3), *and to all* $\Omega(x)$ *obeying the consistency condition* (3.4), with a given $z_{\mu\nu}$. Let $F[\Phi] = 0$ be a gauge-fixing condition. Then, following the standard Faddeev-Popov procedure, we insert the identity

$$1 = \int [d\Omega] \delta[F[\Omega\Phi]] \det \frac{\delta F[\Omega\Phi]}{\delta\Omega}, \tag{4.2}$$

into the functional integral (3.1). The functional integration here extends to all gauge transformations (periodic *and* non-periodic). Using the gauge invariance of the classical action and integration measure[7] one finds

$$\begin{aligned} W[\hat{\mathbf{k}}, \hat{\mathbf{m}}] &= \int [d\Phi][d\Omega] \delta[F[\Omega\Phi]] \det \frac{\delta F[\Omega\Phi]}{\delta\Omega} \exp(-\int d^4x L) \\ &= \int [d\Omega] \int' [d\Phi] \delta[F[\Phi]] \det \frac{\delta F[\Phi]}{\delta\Omega} \exp(-\int d^4x L). \end{aligned} \tag{4.3}$$

The functional integral $\int'[d\Phi]$ extends to all Φ obeying $\Phi(x + a_\mu e_\mu) = (-)^F \Omega'_\mu(x)\Phi(x)$, where

$$\Omega'_\mu(x) = \Omega(x + a_\mu e_\mu)\Omega_\mu(x)\Omega^{-1}(x), \quad \text{no sum on } \mu. \tag{4.4}$$

It is immediate to prove that $\Omega'_\mu(x)$ obeys the same consistency conditions as $\Omega_\mu(x)$, namely (3.4) with the same $z_{\mu\nu}$. Since in the functional integral (3.1) one sums over *all* $\Omega_\mu(x)$ satisfying eq. (3.4), one may rewrite eq. (4.3) as

$$W[\hat{\mathbf{k}}, \hat{\mathbf{m}}] = \mathcal{V} \int [d\Phi] \delta[F[\Phi]] \det \frac{\delta F[\Phi]}{\delta\Omega} \exp(-\int d^4x L). \tag{4.5}$$

Here $\mathcal{V}$ is the volume of the gauge group, which is thus factored out in the standard fashion, and Φ is integrated over the original range, given by eqs. (3.3), (3.4). Everything works as usual,

[6] The regulator has negative mass dimension: $[\text{mass}]^{-3}$, thus the infrared limit is $M \to \infty$.

[7] The gauge symmetry of $N = 4$ is never anomalous, since all fermions are in the adjoint representation of the gauge group.

except that we must integrate over all transition functions $\Omega_\mu(x)$ satisfying the consistency conditions (3.4)[8].

Next, we want to find a gauge fixing which simplifies as much as possible the computation of the functional integral. Since we expect $N=4$ to be in a Coulomb phase, it make sense to look for a gauge fixing that selects out the fields in the Cartan subalgebra of the gauge group. By denoting with α the roots of Lie G, and with $A_\mu^A(x)$, $A_\mu^\alpha(x)$ the gauge fields in and outside of the Cartan subalgebra, respectively, a possible choice is

$$\begin{aligned}
&(a) \qquad [\partial_\mu - i\alpha_A A_\mu^A(x)]A_\mu^\alpha(x) \equiv D_\mu^{Ab.} A_\mu^\alpha(x) = 0, \quad \alpha \neq 0 \\
&(b) \qquad \partial_\mu A_\mu^A(x) = 0, \quad A = 1,..,r = \text{rank } G.
\end{aligned} \tag{4.6}$$

The gauge fixing (a) reduces the gauge symmetry to $U(1)^r$, whereas (b) fixes this Abelian symmetry. The transition functions obeying the gauge-fixing condition (a) in (4.6) are

$$\Omega_\mu(x) = e^{2\pi i \omega_\mu^A(x) T_A}. \tag{4.7}$$

These transition functions are elements of the Cartan subgroup. Indeed, for a generic A_μ in the gauge (4.6), $D_\mu^{Ab.} A_\mu = 0$ together with $D_\mu^{Ab.} \Omega A_\mu = 0$ implies $D_\mu^{Ab.}\Omega = 0$, whose solution is (4.7), again, for generic connections. The Abelian gauge fixing (b) in (4.6), and $A_\mu^A(x + a_\nu e_\nu) = 2\pi\partial_\mu \omega_\nu^A(x) + A_\mu^A(x)$ imply

$$\partial_\mu\partial_\mu \omega_\nu^A(x) = [\partial_\mu A_\mu^A(x + a_\nu e_\nu) - \partial_\mu A_\mu^A(x)]/2\pi = 0, \qquad \text{no sum on } \nu. \tag{4.8}$$

On the $\omega_\mu^A(x)$, the consistency conditions (3.4) become:

$$\omega_\mu^A(x + a_\nu e_\nu) + \omega_\nu^A(x) = \omega_\nu^A(x + a_\mu e_\mu) + \omega_\mu^A(x) + \hat{w}_{\mu\nu}^A, \ \text{ no sum on } \mu,\ \nu. \tag{4.9}$$

Notice that the definition of $\hat{w}_{\mu\nu}$, given in eq. (3.5), is not unique. One can always add to $\hat{w}_{\mu\nu} \in \hat{\Lambda}_W$ a root $\hat{r}_{\mu\nu} \in \hat{\Lambda}_R$. This fact will play an essential role later on. The general solution of eq. (4.9) is

$$\omega_\mu(x) = \sum_\nu \left[\tilde{w}_{\mu\nu}\frac{x_\nu}{a_\nu} + s_{\mu\nu}\frac{x_\nu}{a_\nu}\right] + \tilde{\omega}_\mu(x). \tag{4.10}$$

From now on we shall drop the Cartan index A whenever unambiguous; in this equation

$$\begin{aligned}
\tilde{w}_{\mu\nu} &= \hat{w}_{\mu\nu}, \qquad \text{for} \quad \mu > \nu, \\
\tilde{w}_{\mu\nu} &= 0 \qquad \text{otherwise},
\end{aligned} \tag{4.11}$$

$s_{\mu\nu}^A$ are symmetric coefficients in μ, ν, and $\tilde{\omega}_\mu^A(x)$ is a periodic function. Applying eq. (4.8) to (4.11) gives $\tilde{\omega}_\mu(x) = c_\mu$, where c_μ is a constant and, therefore,

$$\omega_\mu(x) = \sum_\nu \left[\tilde{w}_{\mu\nu}\frac{x_\nu}{a_\nu} + s_{\mu\nu}\frac{x_\nu}{a_\nu}\right] + c_\mu. \tag{4.12}$$

[8]Alternatively, one may pick up a fixed set of transition functions, say $\overset{o}{\Omega}$, and restrict the gauge integration to transformations which commute with $\overset{o}{\Omega}_\mu(x)$. We discard this second possibility for reasons explained after eq. (4.34).

The boundary condition for the gauge field $A^A_\mu(x)$ becomes

$$A^A_\mu(x+a_\nu e_\nu) = A^A_\mu(x) + 2\pi\partial_\mu\omega^A_\nu(x) = A^A_\mu(x) + \frac{2\pi}{a_\mu}\tilde{w}^A_{\nu\mu} + \frac{2\pi}{\alpha_\mu}s^A_{\nu\mu}, \quad \text{no sum on } \nu, \tag{4.13}$$

whose solution reads

$$A_\mu(x) = 2\pi\sum_\nu\left(\tilde{w}_{\nu\mu}\frac{x_\nu}{a_\mu a_\nu} + s_{\mu\nu}\frac{x_\nu}{a_\mu a_\nu}\right) + \tilde{A}_\mu(x). \tag{4.14}$$

where $\tilde{A}_\mu(x)$ is periodic and $\sum_\mu s_{\mu\mu}/a^2_\mu = 0$.

We can get rid of $s_{\mu\nu}$ exploiting the residual invariance of the gauge-fixed functional integral (4.5) under a finite dimensional group of transformations. Any gauge transformation Ω in the Cartan torus such that

$$\Omega(x+a_\mu e_\mu)\Omega_{\omega_\mu}(x)\Omega^{-1}(x) = \Omega_{\omega'_\mu}(x), \tag{4.15}$$

where

$$\Omega_{\omega_\mu}(x) = e^{2\pi i\omega^A_\mu(x)T_A}, \tag{4.16}$$

and both ω_μ and ω'_μ are of the form (4.12), is a symmetry surviving the gauge fixing. These gauge transformations, $\Omega(x) = \exp[2\pi i\omega^A(x)T_A]$, are generated by

$$\omega^A(x) = \beta^A_{\mu\nu}x^\mu x^\nu + \gamma^A_\mu x^\mu + \delta^A, \tag{4.17}$$

with $\beta^A_{\mu\nu}$ a constant, symmetric, traceless matrix, γ^A_μ a constant vector, and δ^A another arbitrary constant. Equation (4.15) is satisfied with

$$\begin{aligned}\omega'_\mu(x) &= \omega_\mu(x) + \beta_{\mu\mu}a^2_\mu + \sum_\nu 2\beta_{\mu\nu}a_\mu a_\nu\frac{x_\nu}{a_\nu} + \gamma_\mu a_\mu \\ &= \sum_\nu\left[\tilde{w}_{\mu\nu}\frac{x_\nu}{a_\nu} + s'_{\mu\nu}\frac{x_\nu}{a_\nu}\right] + c'_\mu.\end{aligned} \tag{4.18}$$

Here

$$s'_{\mu\nu} = s_{\mu\nu} + 2\beta_{\mu\nu}a_\mu a_\nu, \qquad c'_\mu = c_\mu + \beta_{\mu\mu}a^2_\mu + \gamma_\mu a_\mu. \tag{4.19}$$

By choosing $\beta^A_{\mu\nu} = -s^A_{\mu\nu}/2a_\mu a_\nu$, $\gamma^A_\mu = -(c^A_\mu/a_\mu + \beta^A_{\mu\mu}a_\mu)$ we can fix both $s^A_{\mu\nu}$ and c^A_μ to zero.

Now, eq. (4.5) becomes

$$\begin{aligned}W[\hat{\mathbf{k}},\hat{\mathbf{m}}] &= \mathcal{V}\sum_{\hat{w}_{\mu\nu}}\int[d\tilde{A}^A_\mu(x)][dA^\alpha_\mu(x)][d\Phi'(x)][dc(x)][d\bar{c}(x)]\delta[\partial_\mu\tilde{A}^A_\mu(x)]\delta[D^{Ab\cdot}_\mu A^\alpha_\mu(x)] \\ &\quad \exp\left\{-\int d^4x[L+\bar{c}(x)D^{Ab\cdot}_\mu D_\mu c(x)]\right\}.\end{aligned} \tag{4.20}$$

Here we have exponentiated the Jacobian $\delta F[\Phi]/\delta\Omega$ by means of the standard Faddeev-Popov ghosts c, $\bar{c}$. All non-gauge fields (scalars and spinors) are called $\Phi'(x)$, and the sum over $\hat{w}_{\mu\nu}$ is

extended to all vectors in $\hat{\Lambda}_W$ compatible with (3.4). If $\overset{o}{w}_{\mu\nu}$ is one such vector then, as noticed after eq. (4.9), the general $\hat{w}_{\mu\nu}$ reads

$$\hat{w}_{\mu\nu} = \overset{o}{w}_{\mu\nu} + \hat{r}_{\mu\nu}, \qquad \overset{o}{w}_{\mu\nu} \in \hat{\Lambda}_W/\hat{\Lambda}_R, \quad \hat{r}_{\mu\nu} \in \hat{\Lambda}_R. \tag{4.21}$$

Thus the sum in eq. (4.20) runs over all elements $\hat{r}_{\mu\nu}$ of the dual root lattice $\hat{\Lambda}_R$, translated by a fixed dual weight $\overset{o}{w}_{\mu\nu}$. We recall that the relation between $\overset{o}{w}_{\mu\nu}$ and $\hat{\mathbf{m}}, \hat{\mathbf{k}}$ in $W[\hat{\mathbf{k}}, \hat{\mathbf{m}}]$ (4.20) is given in eqs. (3.8), (3.9) with $\hat{w}_{\mu\nu}$ being replaced by $\overset{o}{w}_{\mu\nu}$.

The Abelian field strength $F^A_{\mu\nu} = \partial_\mu A^A_\nu - \partial_\nu A^A_\mu$ reads

$$F^A_{\mu\nu} = \frac{2\pi}{a_\mu a_\nu} \hat{w}^A_{\mu\nu} + \partial_\mu \tilde{A}^A_\nu - \partial_\nu \tilde{A}^A_\mu. \tag{4.22}$$

The boundary conditions of the superpartners of A^A_μ, i.e. the scalar field ϕ^A_{IJ} and the gauginos $\lambda^A_{I\,\alpha}$, are given in eq. (3.3). Explicitly, they read

$$\phi^A_{IJ}(x + a_\mu e_\mu) = \phi^A_{IJ}(x), \quad \lambda^A_{I\,\alpha}(x + a_\mu e_\mu) = -\lambda^A_{I\,\alpha}(x). \tag{4.23}$$

The reason why we have chosen antiperiodic boundary conditions for the fermions along all four directions, in eq. (3.3) and here, is that we want that the functional integrals $W[\hat{\mathbf{k}}, \hat{\mathbf{m}}]$ transform into themselves under discrete 90^0 Euclidean rotations. An example of such rotations, giving rise to 't Hooft duality, is given in Section 5, in equations (5.9), (5.10). Obviously the quantities $w[\hat{\mathbf{k}}, \hat{\mathbf{m}}]$, defined in eq. (4.1), transform under $O(4)$ exactly as $W[\hat{\mathbf{k}}, \hat{\mathbf{m}}]$. Notice that if we had chosen periodic boundary conditions for the fermions, we would have found a topological invariant, the Witten index [23]. This invariant does not depend on the coupling constant nor on theta angle, and thus it is trivially S-duality invariant.

Equations (4.22), (4.23) show that the twist affects only the vector, among the fields of the Cartan supermultiplet. All fields with gauge indices outside of the Cartan subalgebra obey *homogeneous* boundary conditions:

$$\Phi^\alpha(x + a_\mu e_\mu) = (-)^F e^{2\pi i (\omega_\mu(x)\cdot\alpha)} \Phi^\alpha(x), \quad \alpha \in \Lambda_R. \tag{4.24}$$

Notice that $\Phi^\alpha(x) = 0$ satisfies these boundary conditions.

To compute the functional integral (4.20) one must introduce an ultraviolet regulator, for instance a momentum cutoff Λ_0, as well as the infrared regulator $M\beta V$, discussed at the beginning of this Section[9]. We already mentioned that in a finite box the infrared divergences are due to the existence of flat directions in the scalar potential. These cannot be lifted either perturbatively or non-perturbatively [22]. Let us label these flat directions, which can be aligned along the Cartan subalgebra, with v^A_{IJ}, and introduce the following notations:

$$U = \max_{I,J,A} |v^A_{IJ}|, \qquad u = \min_{I,J,A} |v^A_{IJ}|. \tag{4.25}$$

[9]The cutoff Λ_0 may break explicitly gauge invariance. In any non-anomalous gauge theory this breaking can be reabsorbed by adding to the bare action finite non-gauge invariant counterterms. The renormalization approach used here is regularization-independent.

Then the functional integral (4.20) is *defined* as

$$W[\hat{\mathbf{k}}, \hat{\mathbf{m}}] = \mathcal{V} \lim_{M\to\infty} \lim_{\Lambda_0\to\infty} \int_{U\leq M} \prod_{I,J,A} dv_{IJ}^A W_{v_{IJ}^A}[\Lambda_0, \hat{\mathbf{k}}, \hat{\mathbf{m}}], \tag{4.26}$$

$$W_{v_{IJ}^A}[\Lambda_0, \hat{\mathbf{k}}, \hat{\mathbf{m}}] = \sum_{\hat{w}_{\mu\nu}} \int [d\tilde{A}_\mu^A(x)][dA_\mu^\alpha(x)][d\hat{\phi}_{IJ}(x)][d\lambda_I(x)][dc(x)][d\bar{c}(x)]$$
$$\delta[\partial_\mu \tilde{A}_\mu^A(x)]\delta[D_\mu^{Ab\cdot} A_\mu^\alpha(x)] \exp\left\{-\int d^4x [L_{\Lambda_0} + \bar{c}(x) D_\mu^{Ab\cdot} D_\mu c(x)]\right\}, \tag{4.27}$$

where, $\int d^4x \hat{\phi}_{IJ}^A(x) = \beta V v_{IJ}^A$, $\beta = a_4$, $V = a_1 a_2 a_3$. The Lagrangian L_{Λ_0} is regulated in an appropriate manner. A convenient regularization scheme for our purposes is the Wilson-Polchinski one [24, 25, 26, 27]. In this approach, one directly integrates the high-frequency modes in the functional integral, and one defines the renormalization flow by imposing the invariance of the functional integral as the ultraviolet cutoff is changed. This allows us to compute the functional integral (4.20) in two steps, first integrating out the high-energy modes, and then the low-energy ones.

More precisely, one modifies the inverse propagator by cutting out the high-frequency modes. For a scalar of mass m, this is accomplished by introducing a smooth momentum cutoff by the substitution [25]

$$p^2 + m^2 \to K^{-1}(p^2/\Lambda^2)(p^2 + m^2),$$
$$K(p^2/\Lambda^2) = 1,\ p^2 \leq \Lambda^2 - \epsilon,\quad K(p^2/\Lambda^2) = 0,\ p^2 \geq \Lambda^2 + \epsilon,\quad \epsilon > 0. \tag{4.28}$$

This substitution eliminates all modes of frequency larger than $\sqrt{\Lambda^2 + \epsilon}$. The regulated $N = 4$ Lagrangian is then defined as follows: $L_{\Lambda_0} = L_{kin\,\Lambda_0} + L_{int\,\Lambda_0}$. The interaction Lagrangian $L_{int\,\Lambda_0}$ contains all terms of order three or higher in the fields. The kinetic action, $\mathcal{S}_{kin} = \int d^4x L_{kin}$ is quadratic in the fields and reads, in the momentum representation,

$$\mathcal{S}_{kin\,\Lambda_0} = \frac{1}{4\pi} \mathrm{Re}\, S \int \frac{d^4p}{(2\pi)^4} \Phi^*(p) \Pi(p) K^{-1}(p^2/\Lambda_0) \Phi(p). \tag{4.29}$$

$\Phi(p)$ denotes all the $N = 4$ fields, as well as the gauge-fixing ghosts in the momentum representation, whereas their unregulated kinetic term is denoted by $\Pi(p)$. By varying the cutoff from Λ_0 to $\Lambda < \Lambda_0$, one "integrates out" the modes of frequency $\Lambda < \omega < \Lambda_0$, to obtain an effective action at the lower scale Λ. This is the *Wilsonian* effective action

$$\mathcal{S}_\Lambda = \frac{1}{4\pi} \mathrm{Re}\, S \int \frac{d^4p}{(2\pi)^4} \Phi^*(p) \Pi(p) K^{-1}(p^2/\Lambda) \Phi(p) + \int d^4x L_{int\,\Lambda}. \tag{4.30}$$

The interaction Lagrangian at scale Λ depends on $L_{int\,\Lambda_0}$; it may contain a constant term independent of the fields, and it is determined by imposing the equation

$$\Lambda \frac{\partial}{\partial\Lambda} W_{v_{IJ}^A}[\Lambda, \hat{\mathbf{k}}, \hat{\mathbf{m}}] = 0, \tag{4.31}$$

where

$$W_{v_{IJ}^A}[\Lambda, \hat{\mathbf{k}}, \hat{\mathbf{m}}] = \sum_{\hat{w}_{\mu\nu}} \int [d\tilde{A}_\mu^A(x)][dA_\mu^\alpha(x)][d\hat{\phi}_{IJ}(x)][d\lambda_I(x)][dc(x)][d\bar{c}(x)]$$
$$\delta[\partial_\mu \tilde{A}_\mu^A(x)]\delta[D_\mu^{Ab.} A_\mu^\alpha(x)] \exp(-\mathcal{S}_\Lambda). \quad (4.32)$$

By definition $L_{int\,\Lambda_0}$ is the bare interaction Lagrangian [25], given in eq. (2.2). Equation (4.32) means that the functional integral (4.27) can be computed using the low-energy effective action S_Λ, which, by construction, contains only the degrees of freedom of frequency less than Λ. The effective interaction Lagrangian L_Λ at the scale Λ contains relevant operators (of dimension three and four) and irrelevant ones, of dimension higher than four. By fixing the coefficients of these relevant operators, (i.e. the renormalized coupling constants), one defines implicitly the bare coupling constants at scale Λ_0. Renormalizability means that the limit $\Lambda_0 \to \infty$, holding fixed the renormalized coupling constants, exists and is unique. Polchinski [25] proved that this definition of renormalizability coincides with the usual one. Thanks to the finiteness of $N = 4$,[10] a stronger statement holds in our case: the limit $\Lambda_0 \to \infty$ exists at fixed $S = 4\pi/g^2 + i\theta/2\pi$, i.e. at fixed *bare* coupling constant.

The advantage of eq. (4.32) over eq. (4.27) is that $\mathcal{S}_\Lambda$ is much simpler than the original $N = 4$ action. Indeed, $\mathcal{S}_\Lambda$ can be expanded in local operators when $v_{IJ}^A \neq 0$, and it reads

$$\mathcal{S}_\Lambda = \int d^4x \{C[S(\Lambda), \hat{w}_{\mu\nu}] + L_{N=4}[S(\Lambda)] + \Lambda^{-1} L^{(5)} + ...\}. \quad (4.33)$$

Here $L_{N=4}[S(\Lambda)]$ is the dimension ≤ 4 Lagrangian. It is uniquely fixed by $N = 4$ supersymmetry. The only freedom is the choice of the effective coupling constant and theta angle at the scale Λ: $S(\Lambda) = 4\pi/g^2(\Lambda) + i\theta(\Lambda)/2\pi$. The higher-dimension terms ($L^{(5)}$ etc.) are fixed by the renormalization conditions, or equivalently by the bare coupling constants, thanks to the finiteness of $N = 4$. The field-independent constant $C[S(\Lambda), \hat{w}_{\mu\nu}]$ may depend in principle on the fluxes $\hat{w}_{\mu\nu}$.[11]

A major simplification occurs if one chooses $\Lambda \ll u$, where u is defined in eq. (4.25). In this case one may use the fact that all fields outside the Cartan subalgebra have masses $O(gu)$, and use the decoupling theorem [29]. This theorem was proven in the contest of the Wilson-Polchinski scheme in [30], and it states that the effective action at $\Lambda \ll u$ is

$$\mathcal{S}_\Lambda = \int \{C[S^{eff}, \hat{w}_{\mu\nu}] + L_{N=4}^{Ab.}[S^{eff}]\} + O(u^{-\varepsilon}), \quad \varepsilon > 0. \quad (4.34)$$

The Lagrangian $L_{N=4}^{Ab.}$ is the (gauge fixed) $N = 4$ one, with all fields outside the Cartan subalgebra set to zero; thus, it is free and quadratic, since an Abelian $N = 4$ theory is noninteracting. Equation (4.34) gives a precise meaning to the physical intuition that very heavy fields do not contribute to the long range (i.e. low energy) dynamics.

[10] See [28] for a scheme-independent proof and a review of the literature about the problem.

[11] Recall that our boundary conditions (3.3) explicitly break supersymmetry.

The constant $C[S^{eff},\hat{w}_{\mu\nu}]$ arises from two sources. The first is the shift in the vacuum energy due to integrations over the high-energy modes. The flux-dependent part of this contribution comes from the charged modes, outside the Cartan subalgebra (see eq. (4.24)), since the neutral modes $\tilde{A}^A_\mu$, $\hat{\phi}^A_{IJ}$, $\lambda^A_{I\alpha}$ in the Cartan subalgebra do not depend at all on the fluxes $\hat{w}_{\mu\nu}$. The charged modes are massive $m\geq g^{eff}u\equiv(4\pi\mathrm{Re}\,S^{eff})^{-1/2}u$, therefore, they produce contributions to the vacuum energy, whose boundary-dependent part vanishes as $mV^{1/3}\rightarrow\infty$ ($V^{1/3}$=size of the box). Moreover, to compute the functional integral, one must expand around *all* local bosonic minima of the action (i.e. solutions of the classical equations of motion with finite action), and sum over all these minima. The expansion begins with a field-independent term: the action computed on the classical configuration, which obviously contributes to $C[S^{eff},\hat{w}_{\mu\nu}]$. Actually $L^{Ab.}_{N=4}$ already takes into account the contribution of purely Abelian configurations. If we integrate over *all* $\Omega(x)$ obeying the consistency condition (3.4) any non-Abelian configuration necessarily involves (non-gauge) excitations of physical massive modes, since otherwise one could find a gauge transformation bringing it back to an Abelian form. To see whether non-Abelian configurations contribute to the functional integral let us reason as follows. The classical action of $N=4$ is conformally invariant. Thus we may rescale coordinates and fields and express the bosonic action in terms of dimensionless variables $x_\mu=y_\mu/u$, $\phi^a_{IJ}(x)\rightarrow u^{-1}\phi^a_{IJ}(y)$, $A^a_\mu(x)\rightarrow u^{-1}A^a_\mu(y)$. By construction, now $\min_{I,J,A}|v^A_{IJ}|=1$ and

$$\int_{a_\mu} d^4x L_{N=4}[u,\phi^a_{IJ}(x),A^a_\mu(x)]=\int_{ua_\mu} d^4y L_{N=4}[1,\phi^a_{IJ}(y),A^a_\mu(y)]. \tag{4.35}$$

The box, in the y variables, has sizes ua_μ. The classical action $\mathcal{S}_{N=4}$ contributes to the functional integral terms $O[\exp(-\mathcal{S}_{N=4})]$. We want to see what happens in the limit $u\rightarrow\infty$, since our aim is to compute the leading term in the expansion (4.1). Thus, the only non-zero contributions are those whose action is finite in the infinite-volume limit $ua_\mu\rightarrow\infty$. The $N=4$ Lagrangian is given in eq. (2.2). It is immediate to see that under the rescaling $\phi^a_{IJ}(y)\rightarrow\phi^a_{IJ}(\lambda y)$,[12] $A^a_\mu(y)\rightarrow\lambda^{-1}A^a_\mu(\lambda y)$, the bosonic infinite-volume action scales as

$$\begin{aligned}\mathcal{S}_{N=4} &\rightarrow \lambda^{-4}\frac{\mathrm{Re}\,S}{4\pi}\int d^4yV[\phi^a_{IJ}(y)]+\lambda^{-2}\frac{\mathrm{Re}\,S}{4\pi}\int d^4yD_\mu\phi^{a\,IJ}(y)D^\mu\phi^a_{IJ}(y)+\\ &\quad\frac{\mathrm{Re}S}{8\pi}\int d^4yF^a_{\mu\nu}(y)F^{a\,\mu\nu}(y)-\frac{i\mathrm{Im}S}{8\pi}\int d^4yF^a_{\mu\nu}(y)\tilde{F}^{a\,\mu\nu}(y),\\ V[\phi^a_{IJ}(y)] &= f_{abc}f_{ade}\phi^b_{IJ}(y)\phi^{c\,JK}(y)\phi^d_{KL}(y)\phi^{e\,LI}(y)\geq 0\end{aligned} \tag{4.36}$$

Demanding as usual that $\lambda=1$ is a stationary point with finite action, we find that both the potential and the kinetic term of the scalars have to vanish, i.e. that the scalar fields are covariantly constant and mutually commuting

$$D_\mu\phi^a_{IJ}(y)=0,\quad f_{abc}\phi^b_{IJ}(y)\phi^{c\,JK}(y)=0. \tag{4.37}$$

This equation implies that the gauge connection is reducible, that is, it belongs to the subgroup of G leaving ϕ^a_{IJ} invariant (i.e. the stabilizer of $\phi^a_{IJ}(y)$). With a gauge transformation, one can

[12] Notice that this rescaling does not change the constant mode of the scalars.

reduce mutually commuting, covariantly constant scalars to $\phi_{IJ}(y) = v_{IJ}$. For almost all v^A_{IJ} the corresponding stabilizer group is the Cartan torus, and the only gauge connections obeying (4.37) are the Abelian ones, which already appear in the Wilsonian effective action (4.34).

To sum up, the contribution of the constant $C[S^{eff}, \hat{w}_{\mu\nu}]$ to the leading infrared divergence in eq. (4.1) does not depend on $\hat{w}_{\mu\nu}$.

It is important to notice that if we choose to fix a particular set of transition functions, $\overset{o}{\omega}_\mu (x)$ ($= \sum_{\mu\geq\nu} \overset{o}{w}_{\mu\nu}\, x^\nu/a_\mu a_\nu$, for instance) and integrate only on gauge transformations $\Omega(x)$ which leave invariant these transition functions, we cannot reduce the computation of the functional integral to the Abelian case. Indeed, we may find non-Abelian classical configurations whose action stays finite in the $u \to \infty$ limit, and which cannot be gauge-transformed into Abelian configurations. Choose for instance periodic boundary conditions: $\overset{o}{\omega}_\mu (x) = 0$. Then the configuration

$$A^A_\mu(x) = 2\pi \sum_\nu \tilde{w}^A_{\mu\nu} \frac{x_\nu}{a_\mu a_\nu}, \quad \tilde{w}^A_{\mu\nu} \in \hat{\Lambda}_R, \quad \tilde{w}^A_{\mu\nu} \neq 0, \tag{4.38}$$

(all other fields $= 0$), has finite action $\mathcal{S} = \pi[\mathrm{Re}\, S\beta V(\hat{w}_{\mu\nu}\cdot\hat{w}_{\mu\nu})/2a^2_\mu a^2_\nu - i\mathrm{Im}\, S\epsilon^{\mu\nu\rho\sigma}(\hat{w}_{\mu\nu}\cdot \hat{w}_{\rho\sigma})/4]$. Equation (4.38) does not satisfy the boundary conditions $\overset{o}{\omega}_\mu (x) = 0$, but it can be changed into a configuration which does it by a *non-periodic* gauge transformation. This gauge transformation is non-Abelian, i.e. it involves fields outside of the Cartan subalgebra. No *periodic* $\Omega(x)$ exists that brings back the gauge transformed configuration into the Cartan subalgebra. In other words, the scaling argument given above still works, but the scalars do not have a well defined limit at infinity. The gauge transformation needed to bring the reducible connection into the Cartan subalgebra does not belong to the subspace of admissible transformations. By summing over all gauge transformations, i.e. over $r_{\mu\nu} \in \hat{\Lambda}_R$, we enlarge the space of allowed transformations and thus we *can* bring all reducible connections to the Cartan subalgebra.

The extra terms in eq. (4.34) vanish in the limit $u \to \infty$. The complexified, effective coupling constant S^{eff} is a function of the bare coupling constant, and, in general, it may depend on the VEVs v^A_{IJ}, as well as on Λ and the renormalization scale μ: $S^{eff} = f(\Lambda, v^A_{IJ}, \mu, S)$. Here, since the low-energy theory is free, S^{eff} is independent of Λ (recall that $\Lambda \ll u \equiv \min_{IJA} |v^A_{IJ}|$). By dimensional reasons it can only depend on dimensionless ratios $S^{eff} = f(v^A_{IJ}/\mu, S)$. Since the beta function of $N = 4$ is equal to zero, S^{eff} is independent of μ, and thus of v^A_{IJ}: $S^{eff} = f(S)$. This last equation defines S^{eff} in terms of S, and vice-versa[13], thus we may define our $N = 4$ theory using S^{eff} instead of S. This we shall do, and from now on drop the superscript in S^{eff}. In other words, from now on the coupling constant and theta angle will denote the effective, infrared ones.

Finally, once the ultraviolet regulator Λ_0 is removed, we may compute the functional integral (4.26) as follows. First we split the integration over v^A_{IJ} in two parts

$$W[\hat{\mathbf{k}}, \hat{\mathbf{m}}] = \mathcal{V} \lim_{M\to\infty} \left\{ \int_{U\leq M,\, u\geq Q} \prod_{I,J,A} dv^A_{IJ} W_{v^A_{IJ}}[\hat{\mathbf{k}}, \hat{\mathbf{m}}] + \int_{U\leq M,\, u\leq Q} \prod_{I,J,A} dv^A_{IJ} W_{v^A_{IJ}}[\hat{\mathbf{k}}, \hat{\mathbf{m}}] \right\}. \tag{4.39}$$

[13] $f(S)$ is invertible, at least in perturbation theory.

Then we notice that $W[\hat{\mathbf{k}},\hat{\mathbf{m}}]$ is divergent, but the coefficient of the leading divergence, $w[\hat{\mathbf{k}},\hat{\mathbf{m}}]$, is well defined. It is also gauge invariant, and thus it is a good "observable." Moreover, the integral over $u \leq Q$ does not contribute to the leading infrared divergence, as long as $\lim_{M\to\infty} Q/M = 0$. We may choose for instance $Q = \log(M/\mu)$, with μ an arbitrary constant. Now we may use eqs. (4.34), (4.32) to get

$$\begin{aligned} W_{v^A_{IJ}}[\hat{\mathbf{k}},\hat{\mathbf{m}}] &= K[S]\sum_{\hat{w}_{\mu\nu}}\int [d\tilde{A}^A_\mu(x)][dA^\alpha_\mu(x)][d\hat{\phi}_{IJ}(x)][d\lambda_I(x)][dc(x)][d\bar{c}(x)] \\ &\quad \delta[\partial_\mu \tilde{A}^A_\mu(x)]\delta[D^{Ab.}_\mu A^\alpha_\mu(x)]\exp[-\int d^4x L^{Ab.}_{N=4}(S)][1+O(Q^{-\varepsilon})]. \end{aligned} \tag{4.40}$$

$w[\hat{\mathbf{k}},\hat{\mathbf{m}}]$ is the $Q\to\infty$ limit of this equation. The coefficient in front of (4.40) may depend on the coupling constant but is flux-independent. Since the functional integral in eq. (4.40) is quadratic, it can be performed explicitly. The fluctuations about the classical action modify the unknown constant $K[S]$, but they do not introduce any dependence on $\hat{w}_{\mu\nu}$. Thus the overall scale in front of $W[\hat{\mathbf{k}},\hat{\mathbf{m}}]$ is flux-independent and can always be redefined by shifting the vacuum energy by a flux-independent constant. The only nonzero term in the classical action is the field strength (4.22) with $\tilde{A}^A_\mu = 0$, and one finds

$$w[\hat{\mathbf{k}},\hat{\mathbf{m}}] = \sum_{\hat{w}_{\mu\nu}} K'[S]\exp\left[-\pi\sum_{\mu\nu}\left(\beta V\frac{\operatorname{Re} S}{2}\frac{(\hat{w}_{\mu\nu}\cdot\hat{w}_{\mu\nu})}{a^2_\mu a^2_\nu} - i\frac{\operatorname{Im} S}{4}\epsilon^{\mu\nu\rho\sigma}(\hat{w}_{\mu\nu}\cdot\hat{w}_{\rho\sigma})\right)\right]. \tag{4.41}$$

Since the normalization constant $K'[S]$ is a matter of convention, we may choose it, in particular, to be:

$$K'[S] = \left(\frac{V(\operatorname{Re} S)^3}{\beta^3}\right)^{r/2}, \quad r = \operatorname{rank} G. \tag{4.42}$$

Equation (4.41) gives us another test of S-duality, since $w[\hat{\mathbf{k}},\hat{\mathbf{m}}]$ transforms under it as $W[\hat{\mathbf{k}},\hat{\mathbf{m}}]$. In Section 4.2 we shall derive the free energy from $w[\hat{\mathbf{k}},\hat{\mathbf{m}}]$. Its properties under S-duality are examined in Section 6, while factorization, Witten phenomenon, and its transformation laws under 't Hooft's duality are discussed in Section 5.

Before coming to that, the following topics need to be discussed.

- Since the functional integral (3.1) is integrated over all v^A_{IJ}, it corresponds to a mixed thermodynamical phase: the pure phases, in our gauge (4.6), are labelled by the VEVs $\langle\phi^A_{IJ}\rangle$. Moreover, only large VEVs contribute to eq. (4.41). In other words, $w[\hat{\mathbf{k}},\hat{\mathbf{m}}]$ is "blind" to the phase diagram of $N=4$. One cannot argue from eq. (4.41) whether at $\langle\phi^A_{IJ}\rangle = 0$ the theory is in a non-Abelian Coulomb phase. Conformal invariance at zero VEVs, though, together with the fact that $N=4$ is in a Coulomb phase for all non-zero VEVs, makes this hypothesis very plausible.

- In writing the integral (4.39) we did not take into account that some VEVs v^A_{IJ} are identified under a gauge transformation: the gauge fixing we chose still allows for constant gauge transformations in the Weyl group of Lie G. This is a discrete group of finite order,

say $|W|$. Thus, given a $W_{v^A_{IJ}}[\hat{\mathbf{k}}, \hat{\mathbf{m}}]$, we should have divided it by $d(v^A_{IJ})$, that is by the order of the orbit of v^A_{IJ} under the Weyl group, to avoid overcounting its contribution to the functional integral. This proviso is irrelevant for our purpose, which is to compute $w[\hat{\mathbf{k}}, \hat{\mathbf{m}}]$, since for large v^A_{IJ} almost all orbits have dimension $|W|$. Thus we may simply absorb $|W|$ in the definition of the partition function.

- Finally, we may comment on what happens in $N < 4$. One may think that we never used in a fundamental way the $N = 4$ supersymmetry. Indeed we needed it to ensure the vanishing of the beta function. If $\beta \neq 0$, the complexified coupling constant S would depend on v^A_{IJ} in a very non-trivial way [8, 31]; in other words, the scalar potential would receive radiative corrections, both at one loop and at the non-perturbative level. The computation of the integral over v^A_{IJ} would be ill-defined, since S has a logarithmic singularity at $v^A_{IJ} \to \infty$. Moreover, the integration over VEVs would wash out the contribution of the most interesting points in the VEV space, namely those where S is singular, and extra massless degrees of freedom (monopoles and dyons) appear in the low-energy effective action. In the $N = 4$ case, there is no point in looking for a standard QFT description of the theory (with asymptotic particle states etc.) at the point $v^A_{IJ} = 0$. Indeed, at that singular point, there exist both massless electrically charged particles *and* massless monopoles. This means that no local QFT can describe the whole theory. The same conclusion arises if one notices that conformally invariant theories do not have well defined asymptotic states. At $v_{IJ} = 0$ one should rather describe the theory using 4-D superconformal invariance [32]. On the other hand, $N = 2, 1$ theories at their singular points can still be described by a local QFT by adding the appropriate massless degrees of freedom.

4.2 From the Twisted Functional Integral to the Free Energy

We are now ready to write down the leading infrared-divergent contribution to the free energy, which can be derived by using eq. (3.10). This equation says that the free energy is the discrete Fourier transform of the twisted functional integrals; thus its leading infrared divergence is the transform of $w[\hat{\mathbf{k}}, \hat{\mathbf{m}}]$, given in eq. (4.41). Then we will perform a Poisson resummation in the vectors $\hat{w}_{4i}$ to express the free energy in a form verifying and generalizing the suggestion of ref. [17] to any compact, simple group.

From eqs. (3.8), (3.9) and (4.21) we have

$$\hat{w}_{ij} = \epsilon_{ijk}(\hat{l}_k + \hat{m}_k), \qquad \hat{w}_{4i} = \hat{n}_i + \hat{k}_i, \qquad i,j,k = 1,2,3, \qquad \hat{k}_i, \hat{m}_i \in \hat{\Lambda}_W/\hat{\Lambda}_R, \;\; \hat{l}_i, \hat{n}_i \in \hat{\Lambda}_R. \tag{4.43}$$

Inserting into eqs. (4.41), (4.42) one finds

$$w[\hat{\mathbf{k}}, \hat{\mathbf{m}}] = \left(\frac{V}{\alpha^3\beta^3}\right)^{r/2} \sum_{\hat{l}_i, \hat{n}_i \in \hat{\Lambda}_R} e^{-S[\hat{l}_i + \hat{m}_i, \hat{n}_i + \hat{k}_i]}, \tag{4.44}$$

where

$$S[\hat{l}_i+\hat{m}_i,\hat{n}_i+\hat{k}_i]=\pi\sum_{i=1}^{3}\left(\alpha^{-1}\left[\beta_i(\hat{l}_i+\hat{m}_i)^2+\beta_i^{-1}(\hat{n}_i+\hat{k}_i)^2\right]-2ia(\hat{l}_i+\hat{m}_i)(\hat{n}_i+\hat{k}_i)\right), \quad (4.45)$$

(recall (2.3) that $\mathrm{Re}\, S=\alpha^{-1}$, $\mathrm{Im}\, S=a$). In eq. (4.45) and from now on, we define

$$\beta_i=\beta\frac{a_i^2}{V}. \quad (4.46)$$

By using eq. (3.10) we find that the leading infrared-divergent contribution to the free energy is

$$\begin{aligned} e^{-\beta F[\mathbf{e},\hat{\mathbf{m}}]} &= \frac{1}{N^3}\sum_{\hat{\mathbf{k}}\in(\hat{\Lambda}_W/\hat{\Lambda}_R)^3} e^{2\pi i\mathbf{e}\cdot\hat{\mathbf{k}}}w[\hat{\mathbf{k}},\hat{\mathbf{m}}] \\ &= \frac{1}{N^3}\left(\frac{V}{\alpha^3\beta^3}\right)^{r/2}\sum_{\hat{\mathbf{l}}\in\hat{\Lambda}_R^3} e^{-\pi\alpha^{-1}\sum_i\beta_i(\hat{l}_i+\hat{m}_i)^2}\times \\ &\times \sum_{\hat{\mathbf{k}}\in(\hat{\Lambda}_W/\hat{\Lambda}_R)^3}\sum_{\hat{\mathbf{n}}\in\hat{\Lambda}_R^3} e^{2\pi i[\mathbf{e}\cdot(\hat{\mathbf{n}}+\hat{\mathbf{k}})+a(\hat{\mathbf{l}}+\hat{\mathbf{m}})\cdot(\hat{\mathbf{n}}+\hat{\mathbf{k}})]}e^{-\pi\alpha^{-1}\sum_i\beta_i^{-1}(\hat{n}_i+\hat{k}_i)^2}. \end{aligned} \quad (4.47)$$

Here we have inserted $\exp(2\pi i\mathbf{e}\cdot\hat{\mathbf{n}})=1$ inside the sum; this identity holds because $\hat{\Lambda}_R$ is the lattice dual to Λ_W, and because $\mathbf{e}\in\Lambda_W^3$, $\hat{\mathbf{n}}\in\hat{\Lambda}_R^3$ (see Appendix A). Now, we rewrite

$$\hat{w}_i=\hat{n}_i+\hat{k}_i\in\hat{\Lambda}_W \quad (4.48)$$

(this is what we previously called $\hat{w}_{4i}$; here we drop the label 4), and since the elements inside the sum depend on $\hat{n}_i$ and $\hat{k}_i$ only through their sum, $\hat{w}_i$, we may write the double sum over $\hat{\mathbf{k}}\in(\hat{\Lambda}_W/\hat{\Lambda}_R)^3$ and $\hat{\mathbf{n}}\in\hat{\Lambda}_R^3$ as a sum over $\hat{\mathbf{w}}$. We get

$$e^{-\beta F[\mathbf{e},\hat{\mathbf{m}}]}=\frac{1}{N^3}\left(\frac{V}{\alpha^3\beta^3}\right)^{r/2}\sum_{\hat{\mathbf{l}}\in\hat{\Lambda}_R^3} e^{-\pi\alpha^{-1}\sum_i\beta_i(\hat{l}_i+\hat{m}_i)^2}f(\mathbf{e},\hat{\mathbf{m}},\hat{\mathbf{l}}), \quad (4.49)$$

where

$$f(\mathbf{e},\hat{\mathbf{m}},\hat{\mathbf{l}})=\sum_{\hat{\mathbf{w}}\in\hat{\Lambda}_W^3} e^{2\pi i[\mathbf{e}+a(\hat{\mathbf{l}}+\hat{\mathbf{m}})]\cdot\hat{\mathbf{w}}}e^{-\pi\alpha^{-1}\sum_i\beta_i^{-1}\hat{w}_i^2}. \quad (4.50)$$

Expanding $e_i,\hat{w}_i,\hat{l}_i,\hat{m}_i$ in the basis $e_n^*,\hat{e}_n^*,\hat{e}_n,\hat{e}_n^*$, respectively, (the basis we use are described in Appendix A):

$$\begin{gathered} e_i=E_i^n e_n^*, \qquad \hat{w}_i=K_i^n\hat{e}_n^*, \qquad \hat{l}_i=L_i^n\hat{e}_n, \qquad \hat{m}_i=M_i^n\hat{e}_n^*, \\ E_i^n,K_i^n,L_i^n,M_i^n \in Z, \end{gathered} \quad (4.51)$$

we get

$$f(\mathbf{e},\hat{\mathbf{m}},\hat{\mathbf{l}})=\sum_{K_i^n=-\infty}^{\infty} e^{2\pi i[E_i^t\gamma^t+a(L_i^t+M_i^t\gamma)B]K_i}e^{-\pi\alpha^{-1}\sum_i\beta_i^{-1}K_i^tC^{-1}K_i}. \quad (4.52)$$

Here γ, C, B are the $r \times r$ matrices:

$$\gamma_{nm} = \hat{e}^*_n \cdot e^*_m, \qquad C_{nm} = e_n \cdot e_m, \qquad B_{nm} = \hat{e}_n \cdot \hat{e}^*_m = \frac{1}{2}\hat{e}_n^2 \delta_{nm}, \qquad C^{-1} = \gamma B. \tag{4.53}$$

It is now the time to perform a Poisson resummation on the integers K_i. The Poisson resummation formula is:

$$\sum_{K \in Z^n} e^{-(K+\tau)^t A (K+\tau)} = \frac{\pi^{n/2}}{\sqrt{\det A}} \sum_{K \in Z^n} e^{-\pi^2 K^t A^{-1} K} e^{2\pi i K^t \tau}. \tag{4.54}$$

Here A is an $n \times n$ matrix, K and τ are n-vectors. Using the formula (4.54) in eq. (4.52) one finds

$$f(\mathbf{e}, \hat{\mathbf{m}}, \hat{\mathbf{l}}) = (\det C)^{3/2} \left(\frac{\alpha^3 \beta^3}{V}\right)^{r/2} \sum_{K_i \in Z^r} e^{-\pi\alpha \sum_i \beta_i [K_i^t + E_i^t \gamma^t + a(L_i^t + M_i^t \gamma) B] C [K_i + \gamma E_i + a B^t (L_i + \gamma^t M_i)]}. \tag{4.55}$$

Inserting eq. (4.55) into eq. (4.49) we get

$$\begin{aligned} \exp\{-\beta F[E, M, S, \mathcal{G}]\} \;\; &= \;\; c \prod_{i=1}^{3} \sum_{K_i^n, L_i^n = -\infty}^{\infty} \exp\Big\{ \\ & -\pi\beta_i (K_i^n + E_i^p \gamma_{np}, L_i^n + M_i^p \gamma_{pn}) M(S, \mathcal{G}) \begin{pmatrix} K_i^m + E_i^p \gamma_{mp} \\ L_i^m + M_i^p \gamma_{pm} \end{pmatrix} \Big\}, \end{aligned} \tag{4.56}$$

where

$$M(S, \mathcal{G}) = \alpha \begin{pmatrix} (e_n \cdot e_m) & (e_n \cdot \hat{e}_m) a \\ (\hat{e}_n \cdot e_m) a & (\hat{e}_n \cdot \hat{e}_m)(\alpha^{-2} + a^2) \end{pmatrix}, \tag{4.57}$$

the matrix γ is defined in (4.53) and

$$c = \frac{1}{N^3} (\det C)^{3/2}. \tag{4.58}$$

To write eqs. (4.56), (4.57) we have used the relations:

$$(BC)_{nm} = \hat{e}_n \cdot e_m, \qquad (CB^t)_{nm} = e_n \cdot \hat{e}_m, \qquad (BCB^t)_{nm} = \hat{C}_{nm} = \hat{e}_n \cdot \hat{e}_m. \tag{4.59}$$

Finally, one finds

$$\exp\{-\beta F[\mathbf{e}, \hat{\mathbf{m}}, S]\} = c \prod_{i=1}^{3} \sum_{k_i \in \Lambda_R, \hat{l}_i \in \hat{\Lambda}_R} \exp\Big\{ -\pi\beta_i (k_i + e_i, \hat{l}_i + \hat{m}_i) M(S) \begin{pmatrix} k_i + e_i \\ \hat{l}_i + \hat{m}_i \end{pmatrix} \Big\}, \tag{4.60}$$

recall that $\beta_i = \beta a_i^2 / V$ contain all the information about the sizes of the spatial box, a_i, and the temperature β^{-1}. In eq. (4.60), the factor c is a constant, independent of the fluxes $\mathbf{e}$ and $\hat{\mathbf{m}}$ and independent of S, and $M(S)$ is a 2×2 matrix

$$M(S) = \frac{1}{\operatorname{Re} S} \begin{pmatrix} 1 & \operatorname{Im} S \\ \operatorname{Im} S & |S|^2 \end{pmatrix} = \alpha \begin{pmatrix} 1 & a \\ a & \alpha^{-2} + a^2 \end{pmatrix}. \tag{4.61}$$

If $\mathcal{G}$ is simply-laced then $(e_n \cdot e_m) = C_{nm}$ is the Cartan matrix of the Lie algebra and, moreover,

$$\hat{e}_n = e_n,\ \hat{e}^*_n = e^*_n \Rightarrow e_n \cdot e_m = \hat{e}_n \cdot e_m = e_n \cdot \hat{e}_m = \hat{e}_n \cdot \hat{e}_m = C_{nm},$$
$$\gamma_{nm} = (C^{-1})_{nm}. \tag{4.62}$$

Therefore,

$$\exp\{-\beta F[E, M, S,\ \text{simply}-\text{laced}\ \mathcal{G}]\} = c \prod_{i=1}^{3} \sum_{K_i^n, L_i^n = -\infty}^{\infty} \exp\Big\{$$
$$-\pi\beta_i \Big(K_i^n + (E_i C^{-1})^n, L_i^n + (M_i C^{-1})^n\Big) C_{nm} \otimes M(S) \begin{pmatrix} K_i^m + (C^{-1}E_i)^m \\ L_i^m + (C^{-1}M_i)^m \end{pmatrix} \Big\}. \tag{4.63}$$

It is remarkable that eq. (4.63) is formally equal to the classical piece of a twisted genus-1 string partition function on a toroidal background; the genus-1 modular parameter is S, the target-space background matrix is $C \otimes I_{3\times 3}$, and the twist is (E_i, M_i) [14].

5 Properties of the Free Energy

In this Section we discuss some properties of the free energy:

5.1 Factorization

The free energy in eq. (4.60) factorizes at $\theta = 0$:

$$F[\mathbf{e}, \hat{\mathbf{m}}, g, \theta = 0] = F[\mathbf{e}, 0] + F[0, \hat{\mathbf{m}}] + c, \tag{5.1}$$

where c is independent of the fluxes $\mathbf{e}$ and $\hat{\mathbf{m}}$. Such a factorization is physically quite plausible if one assumes the absence of interference between electric and magnetic fluxes in the limit $a_i, \beta \to \infty$ [18]. This holds not only for Abelian fields, as soon as one assumes that strings occupy only a negligible portion of the total space whereas magnetic fields fill the whole space.

An $N = 4$ supersymmetric Yang-Mills theory is scale invariant in an infinite box. In Section 4 we have isolated the leading infrared-divergent contribution to the free energy and, therefore, we expect the free energy in eq. (4.60) to be scale invariant. Indeed, it is invariant under the scale transformation

$$F[La_i, L\beta] = F[a_i, \beta], \tag{5.2}$$

thus, factorization in a large box implies factorization for any size of the box. Indeed, factorizability at $\theta = 0$ is obtained.

5.2 Witten's Phenomenon

The free energy for non-zero θ is derived from the free energy at $\theta = 0$ by the shift of e_i and k_i in the exponent inside the sum over k_i and $\hat{l}_i$ in eq. (4.60):

$$e_i \to e_i + \frac{\theta}{2\pi}\hat{m}_i, \qquad k_i \to k_i + \frac{\theta}{2\pi}\hat{l}_i, \tag{5.3}$$

Explicitly:

$$\begin{aligned}\exp\{-\beta F[\mathbf{e}, \hat{\mathbf{m}}, S]\} &= c\prod_{i=1}^{3}\sum_{k_i\in\Lambda_R, \hat{l}_i\in\hat{\Lambda}_R}\exp\Big\{ \\ & -\pi\beta_i(k_i + e_i + \frac{\theta}{2\pi}(\hat{l}_i + \hat{m}_i), \hat{l}_i + \hat{m}_i)M(g)\begin{pmatrix} k_i + e_i + \frac{\theta}{2\pi}(\hat{l}_i + \hat{m}_i) \\ \hat{l}_i + \hat{m}_i \end{pmatrix}\Big\},\end{aligned} \tag{5.4}$$

where

$$M(g) = M(S)|_{\theta=0} = \begin{pmatrix} \frac{g^2}{4\pi} & 0 \\ 0 & \frac{4\pi}{g^2} \end{pmatrix}. \tag{5.5}$$

This is the Witten phenomenon [19].

To understand why both the non-Abelian electric fluxes, e_i, and the root lattice vectors, k_i, are shifted when θ is turned on, let us discuss the physical meaning of $k_i \in \Lambda_R$. The difference with the Abelian case is that the non-Abelian electric and magnetic fluxes are defined modulo elements of the root lattices. The physical reason for this indeterminacy is the following [18]. The theory contains elementary massive, charged particle carrying charge α under the Cartan $U(1)^r$; here α is a root of G. By pair-producing n such particles, and letting them wind around the box once in the direction i, before annihilating them, we may change the electric flux along i by $k = n\alpha \in \Lambda_R$. The change in energy due to this process is finite: $g^2n^2(\alpha\cdot\alpha)/a_i$. The same argument applies to the magnetic flux, $\hat{m}_i$ which is changed by $\hat{l}_i \in \hat{\Lambda}_R$, *mutatis mutandis*. Therefore, in a given flux sector, $k_i, \hat{l}_i$, the total electric flux, $e_i + k_i$, is shifted by the total magnetic flux, $\hat{m}_i + \hat{l}_i$, when we turn on theta.

Witten's phenomenon also implies that when $\theta \to \theta + 2\pi$, the free energy of electric flux $\mathbf{e}$ should transform into the free energy of electric flux $\mathbf{e} + \hat{\mathbf{m}}$. It is therefore important that

$$e_i + \hat{m}_i \in \Lambda_W, \qquad k_i + \hat{l}_i \in \Lambda_R. \tag{5.6}$$

Let us check that this is indeed the case. Expanding $e_i \in \Lambda_W$ and $\hat{m}_i \in \hat{\Lambda}_W$ in the basis (A.12) and (A.22), respectively,

$$e_i = E_i^n e_n^*, \qquad \hat{m}_i = M_i^n \hat{e}_n^*, \qquad E_i^n, M_i^n \in Z, \tag{5.7}$$

we get

$$\begin{aligned} e_i = E_i^n e_n^* &\;\to\; e_i + \hat{m}_i = E_i^n e_n^* + M_i^n \hat{e}_n^* \Rightarrow \\ E_i^n e_n^* &\;\to\; \sum_n (E_i^n + M_i^m \hat{e}_m^* \cdot \hat{e}_n) e_n^* = \sum_n [E_i^n + \frac{1}{2}(\hat{e}_n\cdot\hat{e}_n)M_i^n] e_n^*. \end{aligned} \tag{5.8}$$

Here we used eqs. (A.18) and (A.25) of Appendix A. Now, from eq. (A.27) we learn that $E_i^n + \frac{1}{2}(\hat{e}_n \cdot \hat{e}_n) M_i^n \in Z$ and, therefore, $e_i + \hat{m}_i \in \Lambda_W$. Similarly, one can show that $k_i + \hat{l}_i \in \Lambda_R$, and this completes the consistency check for $\theta \to \theta + 2\pi$.

5.3 The 't Hooft Duality

Clearly, the functional integral $W[\hat{\mathbf{k}}, \hat{\mathbf{m}}]$, and the coefficient $w[\hat{\mathbf{k}}, \hat{\mathbf{m}}]$ of the leading infrared divergence, must be invariant under joint rotations of a_μ and $\hat{w}_{\mu\nu}$ in Euclidean space. In particular, if we perform the $SO(4)$ rotation corresponding to the interchange $1 \leftrightarrow 2$, $3 \leftrightarrow 4$, represented by the matrix

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \tag{5.9}$$

and keeping in mind the relation between $\hat{m}_i$, $\hat{k}_i$ and $\hat{w}_{\mu\nu}$ given in eqs. (3.8), (3.9), respectively, we find

$$w[\hat{k}_1, \hat{k}_2, \hat{k}_3, \hat{m}_1, \hat{m}_2, \hat{m}_3; a_1, a_2, a_3, \beta] = w[\hat{m}_1, \hat{m}_2, \hat{k}_3, \hat{k}_1, \hat{k}_2, \hat{m}_3; a_2, a_1, \beta, a_3]. \tag{5.10}$$

Indeed, $w[\hat{\mathbf{k}}, \hat{\mathbf{m}}]$ in eq. (4.41) manifestly obeys (5.10).

The consequence of eq. (5.10) for the free energy is

$$\begin{aligned} &\exp\{-\beta F[e_1, e_2, e_3, \hat{m}_1, \hat{m}_2, \hat{m}_3; a_1, a_2, a_3, \beta]\} = \frac{1}{N^2} \sum_{\hat{k}_1, \hat{k}_2 \in \hat{\Lambda}_W/\hat{\Lambda}_R,\, l_1, l_2 \in \Lambda_W/\Lambda_R} \\ &\exp\{2\pi i(\hat{k}_1 \cdot e_1 + \hat{k}_2 \cdot e_2 - l_1 \cdot \hat{m}_1 - l_2 \cdot \hat{m}_2)\} \exp\{-a_3 F[l_1, l_2, e_3, \hat{k}_1, \hat{k}_2, \hat{m}_3, a_2, a_1, \beta, a_3]\}, \end{aligned} \tag{5.11}$$

where $N = Order(C)$. This is the 't Hooft duality relation [18]. Indeed, the free energy in (4.56) obeys (5.11). To prove it, directly for the free energy, one should perform a Poisson resummation on the directions $i = 1, 2$, as was done in [17]. Obviously, here there is nothing to prove since we have computed the functional integrals w and, therefore, 't Hooft's duality is automatic.

6 S-Duality

The S-duality group is generated by the elements $\mathcal{S}, \mathcal{T}$, acting on S by:

$$\mathcal{S}: \qquad S \to \frac{1}{S}, \tag{6.1}$$

$$\mathcal{T}: \qquad S \to S + i, \tag{6.2}$$

or, as we will show soon, acting on $\mathcal{G}$, $\mathbf{e}$ and $\hat{\mathbf{m}}$ by:

$$\mathcal{S}: \qquad \mathcal{G} \leftrightarrow \hat{\mathcal{G}} \ \text{ together with } \ \mathbf{e} \to \mathbf{m},\ \hat{\mathbf{m}} \to -\hat{\mathbf{e}}, \tag{6.3}$$

$$\mathcal{T}\ :\qquad \mathbf{e}\to\mathbf{e}+\hat{\mathbf{m}}. \tag{6.4}$$

By the interchange of the Lie algebra with its dual algebra, $\mathcal{G}\leftrightarrow\hat{\mathcal{G}}$ in (6.3), we mean: $\Lambda_{R,W}\leftrightarrow\hat{\Lambda}_{R,W}$, *i.e.*, $e_n\leftrightarrow\hat{e}_n$, $e_n^*\leftrightarrow\hat{e}_n^*$. By $\mathbf{e}\to\mathbf{m}$, $\hat{\mathbf{m}}\to-\hat{\mathbf{e}}$ in (6.1) we mean: $E_i^n\to M_i^n$, $M_i^n\to -E_i^n$, where the integers E_i^n, M_i^n are defined in eq. (5.7). The shift of $\mathbf{e}$ by $\hat{\mathbf{m}}$ in (6.4) is consistent, because $e_i+\hat{m}_i\in\Lambda_W$; this was checked in Section 5.2.

The elements $\mathcal{S}$ and $\mathcal{T}$ generate a group isomorphic to $SL(2,Z)$, acting on iS by the fractional linear transformations:

$$iS\to\begin{pmatrix} a & b\\ c & d\end{pmatrix}(iS)=\frac{a(iS)+b}{c(iS)+d}, \tag{6.5}$$

where

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL(2,Z),\qquad a,b,c,d\in Z,\qquad ad-bc=1. \tag{6.6}$$

The elements $\mathcal{S}$ and $\mathcal{T}$ correspond, therefore, to the matrices

$$\mathcal{S}=\begin{pmatrix} 0 & -1\\ 1 & 0\end{pmatrix},\qquad \mathcal{T}=\begin{pmatrix} 1 & -1\\ 0 & 1\end{pmatrix}. \tag{6.7}$$

At $\theta=0$, the duality transformation $\mathcal{S}$ takes α to $1/\alpha$ (namely, $\mathcal{S}\ :\ g\to 4\pi/g$), and it is therefore called "strong-weak coupling duality." Alternatively, it interchanges the electric flux $\mathbf{e}$ with the magnetic flux $\hat{\mathbf{m}}$ and, therefore, it is also called "electric-magnetic duality." The transformation $\mathcal{T}$ simply shifts the theta parameter by 2π: $\mathcal{T}\ :\ \theta\to\theta+2\pi$.

When $\mathcal{G}$ is simply-laced, any element of $SL(2,Z)$ in (6.6) transforms the electric and magnetic fluxes into other permitted fluxes. This is not true, however, if $\mathcal{G}$ is non-simply-laced. Therefore, in the following we shall discuss the simply-laced and the non-simply-laced cases separately, when required.

6.1 The Action of S-Duality on the Free Energy for Simply-Laced $\mathcal{G}$

If $\mathcal{G}$ is simply-laced, the Lie algebra and its dual algebra are equal: $\mathcal{G}=\hat{\mathcal{G}}$, namely, $\Lambda_R=\hat{\Lambda}_R$, $\Lambda_W=\hat{\Lambda}_W$. The free energy $F[E,M,S]$, given in eq. (4.63), transforms covariantly under $SL(2,Z)$ S-duality:

$$F\left[E,M,\frac{1}{i}\frac{a(iS)+b}{c(iS)+d}\right]=F[dE-bM,aM-cE,S]. \tag{6.8}$$

To prove eq. (6.8) we note that S-transformation in eq. (6.5) transforms $M(S)$ (4.61) as follows:

$$M\left(\frac{1}{i}\frac{a(iS)+b}{c(iS)+d}\right)=AM(S)A^t,\qquad A=\begin{pmatrix} d & -c\\ -b & a\end{pmatrix}. \tag{6.9}$$

Let us first show how to get eq. (6.9). Let g be an element of $SL(2,R)$, represented by a 2×2 matrix, and acting on a complex variable, x, by a fractional linear transformation:

$$g=\begin{pmatrix} a & b\\ c & d\end{pmatrix},\qquad a,b,c,d\ \in\ R,\qquad ad-bc=1,$$

$$g(x)=\frac{ax+b}{cx+d},\qquad x\ \in\ C. \tag{6.10}$$

Let g_S be the element

$$g_S = \begin{pmatrix} \alpha^{-1/2} & -a\alpha^{1/2} \\ 0 & \alpha^{1/2} \end{pmatrix}, \tag{6.11}$$

where S, α and a are given in eq. (2.3). One finds that

$$g_S(i) = i\alpha^{-1} - a = iS, \tag{6.12}$$

and, therefore,

$$\begin{aligned} g_{S'}(i) = iS' \quad &\equiv \quad \frac{a(iS)+b}{c(iS)+d} = g(iS) = g(g_S(i)) = (gg_S)(i) \\ &\Rightarrow \quad g_{S'} = gg_S. \end{aligned} \tag{6.13}$$

Moreover, one finds that

$$M(S) = \epsilon g_S g_S^t \epsilon^t, \qquad \epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{6.14}$$

where $M(S)$ is given in (4.61). Now, by using eqs. (6.14) and (6.13) we find

$$M(S') = \epsilon g_{S'} g_{S'}^t \epsilon^t = \epsilon g g_S g_S^t g^t \epsilon^t = (\epsilon g \epsilon^t)(\epsilon g_S g_S^t \epsilon^t)(\epsilon g^t \epsilon^t) = AM(S)A^t, \tag{6.15}$$

where

$$A = \epsilon g \epsilon^t = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}. \tag{6.16}$$

With the definition of S' in eq. (6.13), this completes the proof of eq. (6.9). Finally, by using (6.9) in eq. (4.63) one can derive the result (6.8).

6.2 The Action of the Generators on the Fluxes

Equation (6.9) can be used to prove (6.3) and (6.4). The exponent of the free energy in eq. (4.60) transforms to

$$\begin{aligned} \exp\{-\beta F[\mathbf{e}, \hat{\mathbf{m}}, S]\} \quad &\to \quad \exp\{-\beta F[\mathbf{e}, \hat{\mathbf{m}}, (a(iS)+b)/i(c(iS)+d)]\} \\ &= \quad c \prod_{i=1}^{3} \sum_{k_i \in \Lambda_R, \hat{l}_i \in \hat{\Lambda}_R} \exp\Big\{ \\ &\quad -\pi\beta_i(k_i+e_i, \hat{l}_i+\hat{m}_i) AM(S)A^t \begin{pmatrix} k_i+e_i \\ \hat{l}_i+\hat{m}_i \end{pmatrix} \Big\}, \end{aligned} \tag{6.17}$$

where A is given in eq. (6.9). For the transformation $\mathcal{S}$ in (6.7), $a = d = 0, c = -b = 1$:

$$\mathcal{S} \quad : \quad A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Rightarrow$$

$$\begin{aligned} \exp\{-\beta F[\mathbf{e}, \hat{\mathbf{m}}, 1/S]\} \quad &= \quad c \prod_{i=1}^{3} \sum_{k_i \in \Lambda_R, \hat{l}_i \in \hat{\Lambda}_R} \exp\Big\{ -\pi\beta_i(\hat{l}_i+\hat{m}_i, -k_i-e_i) M(S) \begin{pmatrix} \hat{l}_i+\hat{m}_i \\ -k_i-e_i \end{pmatrix} \Big\} \\ &= \quad c \prod_{i=1}^{3} \sum_{k'_i \in \Lambda_R, \hat{l}_i \in \hat{\Lambda}_R} \exp\Big\{ -\pi\beta_i(\hat{l}_i+\hat{m}_i, k'_i-e_i) M(S) \begin{pmatrix} \hat{l}_i+\hat{m}_i \\ k'_i-e_i \end{pmatrix} \Big\} \\ &= \quad \exp\{-\beta F[\hat{\mathbf{m}}, -\mathbf{e}, S]\}. \end{aligned} \tag{6.18}$$

Here we have used the fact that $k_i \in \Lambda_R \Rightarrow k'_i = -k_i \in \Lambda_R$. The result (6.18) proves eq. (6.3). For the transformation $\mathcal{T}$ in (6.7), $a = -b = d = 1, c = 0$:

$$
\begin{aligned}
\mathcal{T} \quad &: \quad A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \Rightarrow \\
\exp\{-\beta F[\mathbf{e}, \hat{\mathbf{m}}, S+i]\} &= c \prod_{i=1}^{3} \sum_{k_i \in \Lambda_R, \hat{l}_i \in \hat{\Lambda}_R} \exp\Big\{ \\
& -\pi\beta_i (k_i + \hat{l}_i + e_i + \hat{m}_i, \hat{l}_i + \hat{m}_i) M(S) \begin{pmatrix} k_i + \hat{l}_i + e_i + \hat{m}_i \\ \hat{l}_i + \hat{m}_i \end{pmatrix} \Big\} \\
&= c \prod_{i=1}^{3} \sum_{k'_i \in \Lambda_R, \hat{l}_i \in \hat{\Lambda}_R} \exp\Big\{ \\
& -\pi\beta_i (k'_i + e_i + \hat{m}_i, \hat{l}_i + \hat{m}_i) M(S) \begin{pmatrix} k'_i + e_i + \hat{m}_i \\ \hat{l}_i + \hat{m}_i \end{pmatrix} \Big\} \\
&= \exp\{-\beta F[\mathbf{e} + \hat{\mathbf{m}}, \hat{\mathbf{m}}, S]\}. \qquad (6.19)
\end{aligned}
$$

Here we have used the fact that $k_i \in \Lambda_R,\ \hat{l}_i \in \hat{\Lambda}_R \Rightarrow k_i + \hat{l}_i \in \Lambda_R$ (see Section 5.2). The result (6.19) proves eq. (6.4).

Finally, let us emphasize that eqs. (6.18) and (6.19) are correct for any $\mathcal{G}$, either simply-laced or non-simply-laced.

6.3 The Action of S-Duality on the Free Energy for Non-Simply-Laced $\mathcal{G}$

From eqs. (6.18) and (6.19) we read that

$$
\begin{aligned}
\mathcal{S}(F[\mathbf{e}, \hat{\mathbf{m}}, S]) &= F[\hat{\mathbf{m}}, -\mathbf{e}, S], \\
\mathcal{T}(F[\mathbf{e}, \hat{\mathbf{m}}, S]) &= F[\mathbf{e} + \hat{\mathbf{m}}, \hat{\mathbf{m}}, S], \qquad (6.20)
\end{aligned}
$$

no matter whether $\mathcal{G}$ is simply-laced or not. In particular, in both cases,

$$
\mathcal{S}^2(F[\mathbf{e}, \hat{\mathbf{m}}, S]) = F[-\mathbf{e}, -\hat{\mathbf{m}}, S]. \qquad (6.21)
$$

But there is a difference between the simply-laced case and the non-simply-laced case. If $\mathcal{G}$ is *non*-simply-laced, the Lie algebra and its dual algebra are different: $\mathcal{G} = so(2n+1) \Leftrightarrow \hat{\mathcal{G}} = sp(2n)$. In Appendix A it is shown that if the long roots of $\hat{\mathcal{G}}$ are normalized with square length 2 then the long roots of the dual algebra have square length 4, namely, $\hat{\Lambda}_R = \sqrt{2}\Lambda_R(\hat{\mathcal{G}})$, $\hat{\Lambda}_W = \sqrt{2}\Lambda_W(\hat{\mathcal{G}})$. Therefore, if in $\mathcal{S}(F)$ we renormalize the roots such that long roots have square length 2, we obtain:

$$
\mathcal{S}(F[\mathbf{e}, \hat{\mathbf{m}}, S]) = F[\mathbf{e}, \hat{\mathbf{m}}, 1/S] = F[\hat{\mathbf{m}}, -\mathbf{e}, S] = F[\hat{\mathbf{m}}/\sqrt{2}, -\sqrt{2}\mathbf{e}, S/2]. \qquad (6.22)
$$

In $F[\hat{\mathbf{m}}/\sqrt{2}, -\sqrt{2}\mathbf{e}, S/2]$, the sum is over vectors $\hat{k}_i + \hat{m}_i/\sqrt{2}$, where $\hat{k}_i \in \Lambda_R(\hat{\mathcal{G}})$, and over vectors $l_i - \sqrt{2}e_i$, where $l_i \in \hat{\Lambda}_R(\hat{\mathcal{G}})$ (recall that $\Lambda_R(\hat{\mathcal{G}})$ is normalized such that roots have square

length 2 or 1, and $\hat{\Lambda}_R(\hat{\mathcal{G}})$ is normalized such that roots have square length 4 or 2). But with this normalization S is changed to $S/2$, and this is the content of the last equality in eq. (6.22).

It might be helpful to rewrite the transformation of the free energy in its presentation $F[E, M, S, \mathcal{G}]$, given in eq. (4.56); it transforms under $\mathcal{S}$ in (6.7) in the following way:

$$F[E, M, 1/S, \mathcal{G}_{e^2_{long}=2}] = F[M, -E, S, \hat{\mathcal{G}}_{e^2_{long}=4}] = F[M, -E, S/2, \hat{\mathcal{G}}_{e^2_{long}=2}]. \tag{6.23}$$

Here $\mathcal{G}_{e^2_{long}=n}$ describes a Lie algebra $\mathcal{G}$ with long roots normalized to have square length equal to n.

Equation (6.22) is by no means problematic. We do, however, run into a problem if we follow an $\mathcal{S}$ transformation by $\mathcal{T}$. In fact, acting on $F[\mathbf{e}, \hat{\mathbf{m}}, S]$ with $\mathcal{TS}$ is not consistent. By using eqs. (6.18) and (6.19) we get:

$$\mathcal{TS}(F[\mathbf{e}, \hat{\mathbf{m}}, S]) = \mathcal{T}(F[\hat{\mathbf{m}}, -\mathbf{e}, S]) = F[\hat{\mathbf{m}} - \mathbf{e}, -\mathbf{e}, S]. \tag{6.24}$$

But $\hat{\mathbf{m}} - \mathbf{e}$ is not a vector in $\hat{\Lambda}^3_W$ ($\hat{\mathbf{m}} - \mathbf{e} \in \Lambda^3_W$ (see Section 5.2), but if $\mathcal{G}$ is non-simply-laced, Λ_W is not in $\hat{\Lambda}_W$. Explicitly,

$$\hat{m}_i - e_i = M^n_i \hat{e}^*_n - E^n_i e^*_n = \sum_n (M^n_i - E^m_i e^*_m \cdot e_n)\hat{e}^*_n = \sum_n \left[M^n_i - \frac{1}{2}(e_n \cdot e_n)E^n_i\right]\hat{e}^*_n, \tag{6.25}$$

and since $e_n \cdot e_n = 1$ or 2, $M^n_i - \frac{1}{2}(e_n \cdot e_n)E^n_i$ is not necessarily an integer and, therefore, $\hat{m}_i - e_i$ is not in $\hat{\Lambda}_W$. In other words, the normalized flux, $(\hat{\mathbf{m}} - \mathbf{e})/\sqrt{2}$ is not in $\Lambda_W(\hat{\mathcal{G}})$ or, equivalently, the shift $S/2 \to S/2 + i/2$ in $F[\hat{\mathbf{m}}/\sqrt{2}, -\sqrt{2}\mathbf{e}, S/2]$ (6.22) is not a 2π theta shift of the coupling $S/2$.) Therefore, $\hat{\mathbf{m}} - \mathbf{e}$ is an "illegal" electric flux. However, it is consistent to act on $F[\mathbf{e}, \hat{\mathbf{m}}, S]$ with $\mathcal{T}^2\mathcal{S}$, since we get

$$\mathcal{T}^2\mathcal{S}(F[\mathbf{e}, \hat{\mathbf{m}}, S]) = F[\hat{\mathbf{m}} - 2\mathbf{e}, -\mathbf{e}, S]. \tag{6.26}$$

Now $\hat{\mathbf{m}} - 2\mathbf{e} \in \hat{\Lambda}^3_W$, so it is a "legal" electric flux in the $\hat{\mathcal{G}}$ theory. (Equivalently, $S/2 \to (S + 2i)/2 = S/2 + i$ is a 2π theta shift of the coupling $S/2$ in eq. (6.22).)

The transformation of $F[\mathbf{e}, \hat{\mathbf{m}}, S]$ under $\mathcal{T}$ in eq. (6.20) is consistent, of course, because $\mathbf{e} + \hat{\mathbf{m}} \in \Lambda_W$ is a legal flux. It might be helpful to write the transformation of $F[E, M, S, \mathcal{G}]$, given in eq. (4.56), under $\mathcal{T}$ in (6.7); it reads:

$$F[E^n, M^n, S + i, \mathcal{G}] = F\left[E^n + \frac{1}{2}(\hat{e}_n \cdot \hat{e}_n)M^n, M^n, S, \mathcal{G}\right]. \tag{6.27}$$

Recall (A.27) that $\hat{e}_n \cdot \hat{e}_n = 2$ or 4 and, therefore, $E^n + (\hat{e}_n \cdot \hat{e}_n)M^n/2$ is an integer, as required.

To summarize, the elements of $SL(2, Z)$ which transform $F[\mathbf{e}, \hat{\mathbf{m}}, S, \mathcal{G}]$ to $F[\mathbf{e}', \hat{\mathbf{m}}', S, \mathcal{G}]$, where $\mathbf{e}, \mathbf{e}' \in \Lambda^3_W$, $\hat{\mathbf{m}}, \hat{\mathbf{m}}' \in \hat{\Lambda}^3_W$, are generated by $\mathcal{T}$ and $\mathcal{ST}^2\mathcal{S}$. These elements generate a subgroup of $SL(2, Z)$, called $\Gamma_0(2)$. However, when we act on $F[\mathbf{e}, \hat{\mathbf{m}}, S]$ with an element of $SL(2, Z)$ containing an odd number of the generator $\mathcal{S}$, the free energy for the Lie algebra $\mathcal{G}$ is transformed into a free energy for the dual Lie algebra $\hat{\mathcal{G}}$. In the simply-laced case, $\hat{\mathcal{G}} = \mathcal{G}$, but in the non-simply laced case $\hat{\mathcal{G}} \neq \mathcal{G}$. Moreover, in the latter case there are elements of $SL(2, Z)$ that transform $F[\mathbf{e}, \hat{\mathbf{m}}, S]$ to a free energy of "illegal" fluxes; for example, $\mathcal{TS}$ is illegal.

6.4 The Partition Function and Electric-Magnetic Duality

So far the discussion about the free energy in a given flux sector, $F[\mathbf{e}, \hat{\mathbf{m}}]$, did not require the use of the gauge *group*, but only the properties of its Lie algebra. However, given a gauge group, G, not all flux sectors are permitted [4]; the electric fluxes, $\mathbf{e}$, are in the weight lattice of the group G (modulo the root lattice of G):

$$e_i \in \frac{\Lambda_W(G)}{\Lambda_R(G)}, \qquad i = 1, 2, 3, \tag{6.28}$$

and the magnetic fluxes, $\hat{\mathbf{m}}$, are in the weight lattice of the dual group $\hat{G}$ (modulo the root lattice of $\hat{G}$, and up to a normalization if $\mathcal{G}$ is not simply-laced):

$$m_i \in \frac{\hat{\Lambda}_W(G)}{\hat{\Lambda}_R(G)}, \qquad i = 1, 2, 3. \tag{6.29}$$

(We recall that $\hat{\Lambda}(G)_{R,W} = N(\mathcal{G})\Lambda(\hat{G})_{R,W}$, where $N(\mathcal{G}) = 1$ if G is simply-laced, and $N(\mathcal{G}) = \sqrt{2}$ if G is non-simply-laced. In general, $\Lambda_W(G)$ is a sub-lattice of the weight lattice of the Lie algebra $\Lambda_W(\mathcal{G})$, which is the weight lattice of the universal covering group. The weight lattice of the dual group, $\Lambda_W(\hat{G})$ is dual to the weight lattice of G. See Appendix A for more details.)

The partition function for a group G, $\mathcal{Z}(G)$, is given by summing over the allowed flux sectors:

$$\mathcal{Z}(G, S) = \sum_{\mathbf{e} \in (\Lambda_W(G)/\Lambda_R(G))^3,\ \hat{\mathbf{m}} \in (\hat{\Lambda}_W(G)/\hat{\Lambda}_R(G))^3} e^{-\beta F[\mathbf{e}, \hat{\mathbf{m}}, S]}. \tag{6.30}$$

Therefore, under strong-weak coupling duality, $\mathcal{S}$, the partition function transforms to

$$\begin{aligned} \mathcal{S} \ &: \ \mathcal{Z}(G, S) \to \mathcal{Z}(G, 1/S) = \mathcal{Z}(\hat{G}, S), \qquad \text{if} \quad G \text{ simply} - \text{laced} \\ \mathcal{S} \ &: \ \mathcal{Z}(G, S) \to \mathcal{Z}(G, 1/S) = \mathcal{Z}(\hat{G}, S/2), \qquad \text{if } G \text{ non} - \text{simply} - \text{laced}. \end{aligned} \tag{6.31}$$

This equation tells us that *the partition function of an $N = 4$ Yang-Mills theory with a coupling constant $1/S$ and a simply-laced (non-simply-laced) gauge group G is identical to the partition function of a theory with coupling S $(S/2)$ and a dual gauge group $\hat{G}$.*

Moreover, if n is an integer such that $e_i + n\hat{m}_i \in \Lambda_W(G)$ for any $e_i \in \Lambda_W(G)/\Lambda_R(G)$ and $\hat{m}_i \in \hat{\Lambda}_W(G)/\hat{\Lambda}_R(G)$, then the partition function is invariant under $\mathcal{T}^n$:

$$\mathcal{T}^n \ : \qquad \mathcal{Z}(G, S) \to \mathcal{Z}(G, S + ni) = \mathcal{Z}(G, S). \tag{6.32}$$

Therefore, as it will be shown below, the partition function is invariant under the subgroup generated by $\mathcal{T}^n$ and $\mathcal{S}\mathcal{T}^{\hat{n}}\mathcal{S}$. Here $\hat{n}$ is an integer such that $\hat{m}_i + \hat{n}e_i \in \hat{\Lambda}_W(G)$ for any $e_i \in \Lambda_W(G)/\Lambda_R(G)$ and $\hat{m}_i \in \hat{\Lambda}_W(G)/\hat{\Lambda}_R(G)$; for such an $\hat{n}$ the free energy for the dual group $\hat{G}$ is invariant under $\mathcal{T}^{\hat{n}}$:

$$\begin{aligned} \mathcal{Z}(\hat{G}, S + \hat{n}i) &= \mathcal{Z}(\hat{G}, S), \qquad \text{if} \quad G \text{ simply} - \text{laced} \\ \mathcal{Z}(\hat{G}, (S + \hat{n}i)/2) &= \mathcal{Z}(\hat{G}, S/2), \qquad \text{if } G \text{ non} - \text{simply} - \text{laced}. \end{aligned} \tag{6.33}$$

We thus find that

$$\begin{aligned}
\mathcal{S}\mathcal{T}^{\hat{n}}\mathcal{S}(\mathcal{Z}(G,S)) = \mathcal{S}\mathcal{T}^{\hat{n}}(\mathcal{Z}(\hat{G},S)) = \mathcal{S}(\mathcal{Z}(\hat{G},S)) = \mathcal{Z}(G,S), &\quad \text{if} \quad G \text{ simply} - \text{laced} \\
\mathcal{S}\mathcal{T}^{\hat{n}}\mathcal{S}(\mathcal{Z}(G,S)) = \mathcal{S}\mathcal{T}^{\hat{n}}(\mathcal{Z}(\hat{G},S/2)) = \mathcal{S}(\mathcal{Z}(\hat{G},S/2)) = \mathcal{Z}(G,S), &\quad \text{if } G \text{ non} - \text{simply} - \text{laced.}
\end{aligned} \tag{6.34}$$

This shows that $\mathcal{Z}$ is invariant under the subgroup of $SL(2,Z)$ generated by $\mathcal{T}^n$ and $\mathcal{S}\mathcal{T}^{\hat{n}}\mathcal{S}$.

7 Summary and Conclusions

In this paper we have defined some gauge invariant quantities in $N=4$ super Yang-Mills theories based on arbitrary compact, simple groups (the generalization to arbitrary compact groups is straightforward). They are the partition functions with twisted boundary conditions and the corresponding free energies. To define the twist, as well as the non-Abelian equivalent of the electric and magnetic fluxes, we extended straightforwardly the definitions given by 't Hooft for the $SU(N)$ case. The partition function is infrared divergent, and the main result of this paper is that the leading infrared divergence, in the expansion given in eq. (4.1), is exactly computable. Upon performing this computation, we derived the corresponding leading divergence of the free energies in all flux sectors.

We defined the transformation laws under S-duality of the free energies. These laws are general and must hold in any theory $SL(2,Z)$-duality symmetric, and whose transformation laws under S-duality obey the Montonen-Olive conjecture. Finally we verified that these laws are obeyed by the quantities we computed, thereby providing another independent check of the S-duality conjecture in $N=4$ supersymmetric theories.

In studying S-duality on non-simply laced groups, we noticed an interesting phenomenon. Namely, we found in Section 6.3 that there exist $SL(2,Z)$ transformations that are not allowed, since they would transform physical fluxes into unphysical ones. Obviously, one can modify the definition of S-duality so as to cure this problem. This is done by defining the $\mathcal{T}$ transformation as being always a 2π theta shift. Thus, one defines $\mathcal{T}\mathcal{S}$ as (Cfr. eqs. (6.24) and (6.22))

$$\mathcal{T}\mathcal{S}(F[\mathbf{e},\hat{\mathbf{m}},S]) = \mathcal{T}(F[\hat{\mathbf{m}}/\sqrt{2},-\sqrt{2}\mathbf{e},S/2]) \equiv F[\hat{\mathbf{m}}/\sqrt{2},-\sqrt{2}\mathbf{e},S/2+i]. \tag{7.1}$$

Even with this new definition, only a subgroup of all fractional linear transformations $iS' = (aiS+b)/(ciS+d)$, $a,b,c,d \in Z$, $ad-bc=1$ is realized by transformations among physical fluxes. For instance, as we saw in Section 6.3, $iS' = -(1/iS+1)$ does not belong to this subgroup. This means that when S is promoted to a true dynamical field, only a subgroup of $SL(2,Z)$ becomes a true symmetry. It is interesting to notice that this reduction of symmetry happens only in non-simply laced gauge groups, which can never be obtained from $N=4$ supersymmetric compactifications of the heterotic string!

Finally, we should remark that further highly non-trivial tests of S-duality in $N=4$ Yang-Mills theories could be done by computing the subleading terms in the infrared-divergence expansion (4.1).

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Note Added

In two independent papers [33], which appeared during the completion of this manuscript, a reduction of 4-d super YM to supersymmetric 2-d sigma models is considered, leading to a map of S-duality to T-duality of the SCFT. Some of the results of [33] could be related to this work, in particular, to eqs. (4.63) and (B.7).

Appendix A - Notations and Algebra

Let G denote a compact, simple Lie group, and let $\tilde{G}$ be the universal covering group of G (the generalization to semi-simple groups is obvious). The center of G – the set of elements commuting with *all* the group elements – will be denoted by $C(G)$. The center of $\tilde{G}$ will be denoted by $C \equiv C(\tilde{G})$. The group G is related to its universal covering by the quotient:

$$G = \frac{\tilde{G}}{K}, \qquad K \subseteq C. \tag{A.1}$$

The Lie algebra of G will be denoted by $\mathcal{G}$. Obviously, the Lie algebra is the same for any sub-group $K \subseteq C$, because the center of a compact, semi-simple group is discrete. The dual group of G will be denoted by

$$\hat{G} \equiv G^{dual}. \tag{A.2}$$

The dual Lie algebra – the Lie algebra of $\hat{G}$ – will be denoted by

$$\hat{\mathcal{G}} \equiv \mathcal{G}^{dual}. \tag{A.3}$$

The definition of the dual group and the dual algebra will become evident from the discussion below. Here we shall only mention a few properties of the duality operation. First, the dual of a dual group is the original group:

$$dual^2 \equiv (\wedge)^2 = 1. \tag{A.4}$$

The dual Lie algebra equals to the original Lie algebra when the group is simply-laced:

$$G \text{ simply} - \text{laced} \Rightarrow \mathcal{G} = \hat{\mathcal{G}}, \tag{A.5}$$

and for non-simply-laced groups:

$$\mathcal{G} = so(2n+1) \Leftrightarrow \hat{\mathcal{G}} = sp(2n), \qquad \mathcal{G} = \hat{\mathcal{G}} \text{ for } G_2, F_4. \tag{A.6}$$

Next we shall discuss several lattices: the root lattice, the weight lattice and their duals. The root lattice of the Lie algebra $\mathcal{G}$ will be denoted

$$\Lambda_R \equiv \Lambda_{Root}(\mathcal{G}), \tag{A.7}$$

and we denote a basis of the root lattice by

$$\{e_n\}_{n=1,\ldots,r} \in \Lambda_R, \qquad r = \operatorname{rank} G, \tag{A.8}$$

where e_n is a vector with r components: $e_n \equiv (e_n^1, .., e_n^r)$. A convenient choice for a basis is the set of simple roots, normalized such that

$$\begin{aligned} \{e_n\}_{n=1,\ldots,r} &= \{\text{simple roots}\}, \\ e_n^2 &= 2 \qquad \text{for long roots}, \\ (&\Rightarrow e_n^2 = 1 \qquad \text{for short roots}). \end{aligned} \tag{A.9}$$

In this work, we assume this choice, unless otherwise specified. We define the $r \times r$ matrix C by

$$C_{nm} = e_n \cdot e_m \equiv \sum_{P=1}^{r} e_n^P e_m^P. \tag{A.10}$$

C is related to the Cartan matrix $2(e_n \cdot e_m)/e_n^2$; the two are equal if $\mathcal{G}$ is simply-laced.

The weight lattice of the Lie algebra $\mathcal{G}$ will be denoted

$$\Lambda_W \equiv \Lambda_{Weight}(\mathcal{G}), \tag{A.11}$$

and we denote a basis of the weight lattice by

$$\{e_n^*\}_{n=1,\ldots,r} \in \Lambda_W. \tag{A.12}$$

The basis e_n^* can be chosen such that it obeys:

$$\frac{2(e_n \cdot e_m^*)}{(e_n \cdot e_n)} = \delta_{nm}. \tag{A.13}$$

For simply-laced groups, all roots have the same length ($e_n^2 = 2$ for all n) and, therefore, (A.13) implies that

$$\mathcal{G} \text{ simply} - \text{laced} \Rightarrow e_n^2 = 2 \Rightarrow e_n \cdot e_m^* = \delta_{nm}. \tag{A.14}$$

However, for non-simply-laced groups there are long roots ($e_n^2 = 2$) as well as short roots ($e_n^2 = 1$) and, therefore, (A.13) implies that

$$\mathcal{G} \text{ non} - \text{simply} - \text{laced} \Rightarrow e_n^2 = 2 \text{ or } 1 \Rightarrow e_n \cdot e_m^* = \delta_{nm} \text{ or } \frac{1}{2}\delta_{nm}. \tag{A.15}$$

The dual lattice of the weight lattice of $\mathcal{G}$ is equal (up to normalization) to the root lattice of the dual algebra $\hat{\mathcal{G}}$ and, therefore, we will denote it by

$$\hat{\Lambda}_R \equiv (\Lambda_W(\mathcal{G}))^{dual}. \tag{A.16}$$

A basis to $\hat{\Lambda}_R$ will be denoted

$$\{\hat{e}_n\}_{n=1,\dots,r} \in \hat{\Lambda}_R. \tag{A.17}$$

The basis $\hat{e}_n$ can be chosen such that it obeys:

$$\hat{e}_n \cdot e_m^* = \delta_{nm}. \tag{A.18}$$

From eqs. (A.13) and (A.18) it thus follows that

$$\hat{e}_n = \frac{2e_n}{(e_n \cdot e_n)}, \tag{A.19}$$

and from (A.19) one can show that the explicit relation between $\hat{\Lambda}_R$ and the root lattice of the dual algebra is:

$$\begin{aligned} G \text{ simply} - \text{laced} \quad &\Rightarrow \quad \hat{\Lambda}_R = \Lambda_R, \\ G \text{ non} - \text{simply} - \text{laced} \quad &\Rightarrow \quad \hat{\Lambda}_R = \sqrt{2}\Lambda_R(\hat{\mathcal{G}}). \end{aligned} \tag{A.20}$$

The dual lattice of the root lattice of $\mathcal{G}$ is equal (up to normalization) to the weight lattice of the dual algebra $\hat{\mathcal{G}}$ and, therefore, we will denote it by

$$\hat{\Lambda}_W \equiv (\Lambda_R(\mathcal{G}))^{dual}. \tag{A.21}$$

A basis to $\hat{\Lambda}_W$ will be denoted

$$\{\hat{e}_n^*\}_{n=1,\dots,r} \in \hat{\Lambda}_W. \tag{A.22}$$

The basis $\hat{e}_n^*$ can be chosen such that it obeys:

$$\hat{e}_n^* \cdot e_m = \delta_{nm}. \tag{A.23}$$

From eqs. (A.10) and (A.23) it follows that

$$\hat{e}_n^* \cdot \hat{e}_m^* = (C^{-1})_{nm}. \tag{A.24}$$

By using the definition of $\hat{e}_n$ and $\hat{e}_m^*$ and eq. (A.13) one can find that

$$\frac{2(\hat{e}_n \cdot \hat{e}_m^*)}{(\hat{e}_n \cdot \hat{e}_n)} = \delta_{nm} \tag{A.25}$$

From eqs. (A.25) and (A.20) it thus follows that the explicit relation between $\hat{\Lambda}_W$ and the weight lattice of the dual algebra is:

$$\begin{aligned} G \text{ simply}-\text{laced} \quad &\Rightarrow \quad \hat{\Lambda}_W = \Lambda_W, \\ G \text{ non}-\text{simply}-\text{laced} \quad &\Rightarrow \quad \hat{\Lambda}_W = \sqrt{2}\Lambda_W(\hat{\mathcal{G}}). \end{aligned} \tag{A.26}$$

Finally, from eq. (A.19) one finds that

$$\begin{aligned} G \text{ simply}-\text{laced} \quad &\Rightarrow \quad e_n^2 = 2 \Rightarrow \hat{e}_n^2 = 2, \\ G \text{ non}-\text{simply}-\text{laced} \quad &\Rightarrow \quad e_n^2 = 2 \text{ or } 1 \Rightarrow \hat{e}_n^2 = 2 \text{ or } 4. \end{aligned} \tag{A.27}$$

From eqs. (A.25) and (A.27) we thus learn that

$$\begin{aligned} G \text{ simply}-\text{laced} \quad &\Rightarrow \quad \hat{e}_n \cdot \hat{e}_m^* = \delta_{nm}, \\ G \text{ non}-\text{simply}-\text{laced} \quad &\Rightarrow \quad \hat{e}_n \cdot \hat{e}_m^* = \delta_{nm} \text{ or } 2\delta_{nm}. \end{aligned} \tag{A.28}$$

The center, C, of $\tilde{G}$ is:

$$C = \{e^{2\pi i \hat{w}\cdot T} | \hat{w} \in \hat{\Lambda}_W / \hat{\Lambda}_R\}. \tag{A.29}$$

Here $\hat{w}$ is a vector with components $\hat{w}^P$, $P = 1,...,r$, $r = \operatorname{rank} G$, and $\{T_P\}_{P=1,...,r}$ are the generators in the CSA. A weight $w = (w_1,...,w_r)$ is the eigenvalue of $(T_1,...,T_r)$ corresponding to one common eigenvector in a single valued representation of G:

$$T_P V_w = w_P V_w, \qquad w \in \Lambda_W. \tag{A.30}$$

(We should remark that eq. (A.29) is true for any choice of normalization, because (A.30) implies that the normalization of $w \in \Lambda_W$ and $\alpha \in \Lambda_R$ is compatible with the choice of generators T. Thus C is indeed the center, because $w \cdot \hat{\alpha} \in Z$ for any $w \in \Lambda_W$ and $\hat{\alpha} \in \hat{\Lambda}_R = (\Lambda_W)^{dual}$, and this can be used to prove that the elements in (A.29) are the full set of group elements which commute with all the generators.)

For convenience let us summarize our notations and normalizations:

- G is a compact, simple group.
- $\tilde{G}$ is the universal covering group of G.
- $G = \tilde{G}/K$, $K \subseteq C$, $C \equiv Center(\tilde{G})$.
- $\mathcal{G}$ is the Lie algebra of G.
- $\hat{G}$ is the dual group of G.
- $\hat{\mathcal{G}}$ is the dual Lie algebra, *i.e.*, the Lie algebra of $\hat{G}$.
- Λ_R is the root lattice of $\mathcal{G}$, with normalization (long root)$^2 = 2$.
- Λ_W is the weight lattice of $\mathcal{G}$.

- $\hat{\Lambda}_R$ is the dual lattice of Λ_W.
- $\hat{\Lambda}_W$ is the dual lattice of Λ_R.
- $\hat{\Lambda}(\mathcal{G}) = N(\mathcal{G})\Lambda(\hat{\mathcal{G}})$, where $N(\mathcal{G}) = 1$ if G is simply-laced, and $N(\mathcal{G}) = \sqrt{2}$ if G is non-simply-laced; Λ is either Λ_R or Λ_W.
- The group G has a weight lattice of representations which is a sub-lattice of Λ_W: $G = \tilde{G}/K \Rightarrow \Lambda_W(G) = \Lambda_W/K$.
- The dual group $\hat{G}$ has a weight lattice dual to the weight lattice of G: $\Lambda_W(\hat{G}) = \Lambda_W(G)^{dual}$.
- $\hat{\Lambda}(G)_{R,W} = N(\mathcal{G})\Lambda(\hat{G})_{R,W}$, where $N(\mathcal{G}) = 1$ if G is simply-laced, and $N(\mathcal{G}) = \sqrt{2}$ if G is non-simply-laced.
- For G simply-laced: $\hat{\mathcal{G}} = \mathcal{G}$.
- For G non-simply-laced: $\mathcal{G} = so(2n+1) \Leftrightarrow \hat{\mathcal{G}} = sp(2n)$. (The Lie algebrae of G_2 and F_4 are self-dual.)

Appendix B - the Hamiltonian Formalism

The free energy $F[\mathbf{e}, \hat{\mathbf{m}}, S]$ can be calculated, in principle, also in the Hamiltonian formalism [17], and one can check that our result is consistent with the weak coupling Hamiltonian evaluation. The free energy at the different flux sectors reads [18]:

$$e^{-\beta F[\mathbf{e},\hat{\mathbf{m}},S]} = \mathrm{Tr}_{\hat{\mathbf{m}}} P[\mathbf{e}] e^{-\beta H}, \tag{B.1}$$

$$P[\mathbf{e}] = \prod_{i=1}^{3} P[e_i], \qquad P[e_i] = \frac{1}{N} \sum_{\hat{k}_i \in \hat{\Lambda}_W/\hat{\Lambda}_R} \Omega_{\hat{k}_i} e^{2\pi i \hat{k}_i \cdot e_i} \tag{B.2}$$

$(N = Order(C))$,

$$\Omega_{\hat{k}_i} = e^{2\pi i \hat{k}_i \cdot T \frac{x_i}{a_i}} \tag{B.3}$$

(the generators T are in the CSA, see Appendix A). The symbol $\mathrm{Tr}_{\hat{\mathbf{m}}}$ in (B.1) denotes the trace over gauge fields obeying boundary conditions twisted by $\hat{\mathbf{m}}$. $P[\mathbf{e}]$ is the projector onto states with a definite value of the electric fluxes. The gauge transformations $\Omega_{\hat{k}_i}$ in eq. (B.3) are periodic up to a non-trivial element of the center of $\tilde{G}$, $\exp(2\pi i \hat{k}_i \cdot T) \in C$:

$$\begin{aligned} \Omega_{\hat{k}_i}(x_j + a_j) &= e^{2\pi i \hat{k}_i \cdot T} \Omega_{\hat{k}_i}(x_j) && \text{if } j = i, \\ \Omega_{\hat{k}_i}(x_j + a_j) &= \Omega_{\hat{k}_i}(x_j) && \text{if } j \neq i. \end{aligned} \tag{B.4}$$

It must be stressed that the free energy in eq. (B.1) is the finite temperature free energy which assumes antiperiodicity in time for the fermions. Unfortunately, it is complicated to evaluate this trace for every regime. But, as in [17], one can compute its $\theta = \hat{\mathbf{m}} = 0$, $g \ll 1$

limit, finding complete agreement with the leading infrared-divergent part of the free energy, evaluated with the functional integral approach in Section 4. The role of the infrared divergence is crucial to get the correct result for a general compact group [34].[14]

Since the coupling constant is small, we can compute $F[\mathbf{e}, 0, g]$ by using perturbation theory. This possibility exists since $N = 4$ super Yang-Mills has a vanishing beta function; the theory with $g \ll 1$ is weakly interacting at *any* length scale. We set $\hat{\mathbf{m}}$ to zero since in this regime the magnetic flux is expected to interact strongly, with coupling constant $O(4\pi/g) \gg 1$. At $\hat{\mathbf{m}} = 0$ all fields in the box obey periodic boundary conditions, up to a periodic, globally defined gauge transformation. In the box, all their non-zero Fourier modes have energies $O(V^{-1/3})$. Moreover, the potential energy scales as $1/g^2$. For instance, the gauge fields possess a "magnetic" energy $\int d^3x g^{-2}\mathbf{B^2}$. The zero modes belonging to the Cartan subalgebra, instead, have energies $O(g^2 V^{-1/3})$. At $g \ll 1$ only this latter set gives a significant contribution to the free energy [23, 35]. If the spatial supersymmetry is enforced by choosing periodic spatial boundary conditions for fermions, the quantum fluctuations do not destroy the classical degeneracy of the manifold of the gauge zero modes, unlike in the non-supersymmetric case [35]. The low energy spectrum is therefore obtained by quantizing such manifold. The Hamiltonian evaluation with periodic fermions (in space) continues to be a good check at weak coupling of our result since this different choice of boundary conditions does not affect the functional integral evaluation of Section 4.

The configurations contributing to the partition function at $g \ll 1$ are, therefore, the general gauge transformations of the zero modes belonging to the CSA. Notice that the constant gauge configurations, $\mathbf{c}$, are defined only up to periodic gauge transformations. The element

$$\Omega_{\hat{r}_i} = e^{2\pi i \hat{r}_i \cdot T \frac{x_i}{a_i}}, \qquad \hat{r}_i \in \hat{\Lambda}_R, \qquad i = 1, 2, 3 \tag{B.5}$$

is well-defined and periodic in the box in which we have defined the system; thus it generates a gauge transformation, which acts on the zero modes as

$$c_i \longrightarrow c_i + \frac{2\pi}{a_i}\hat{r}_i. \tag{B.6}$$

Therefore, the zero modes c_i are periodic; moreover, we have to mod out by the Weyl group which is realized by other constant gauge transformations. The only periodic gauge transformations that map the Cartan torus to itself are $\Omega_{\hat{r}_i}$ (B.5) and the constant Weyl transformations. Thus the zero modes of the gauge fields parametrize a compact space with boundary, called the toron manifold. Moreover, the Weyl group acts also on constant scalar fields modes in the CSA, ϕ_{IJ}, thus (ϕ_{IJ}, c_i) live on the orbifold

$$(\phi_{IJ}, c_i) \in \frac{R^{6r} \times \prod_{i=1}^{3} T_i^r}{\mathrm{Weyl-Group}}, \qquad T_i^r \equiv \frac{R^r}{2\pi\hat{\Lambda}_R/a_i}, \qquad r = \mathrm{rank}\, G. \tag{B.7}$$

The Lagrangian restricted to the modes constant in space reads

$$L = \frac{V}{g^2}\int dt (\sum_i \dot{c}_i \cdot \dot{c}_i + \dot{a}_I^A \cdot a_I^{A\,\dagger} + \dot{\phi}_{IJ} \cdot \dot{\phi}_{IJ}), \tag{B.8}$$

[14] We thank C. Imbimbo and S. Mukhi for informing us about their results prior to publication.

where c_i, a_I^A, ϕ_{IJ} are the zero modes in the CSA of the gauge fields, fermions and scalars, respectively, and $A = 1, 2$ is the spinor index. In terms of the conjugate momenta, π_i, of the gauge zero modes c_i, the Hamiltonian is

$$H = \frac{g^2}{4V} \sum_i \pi_i \pi_i + scalars. \tag{B.9}$$

Clearly, the eigenfunctions are plane waves,

$$e^{ik_i \cdot c_i} \tag{B.10}$$

(where $k_i \in a_i \Lambda_W$ for ensuring single-valuedness on the compact space of the c_i), with energy,

$$E = \frac{g^2}{4V} \sum_i k_i^2. \tag{B.11}$$

The fermion zero modes do not contribute to the energy, while the scalar zero modes contribute with the energy of a plane wave of continuous momentum.

Due to eq. (B.7), the partition function is not simply the sum of the statistical weights associated with these energies over all the weight lattice (i.e. it is not a theta function), since we have to mod out by the Weyl group. The energy term $(g^2/4V)\mathbf{k}^2$, being the natural Cartan scalar product on the weight lattice, is invariant, but the multiplicity with which we must count the single vector $\mathbf{k}$ depends on the number of invariants under the Weyl group we can construct with the product of the gauge, fermionic and scalar wavefunctions. Since the Weyl group does not act freely on the weight lattice, the correct partition function, in general, is not a theta function (in particular in the purely bosonic case). The exact computation of the multiplicities is a complicated group theory problem; fortunately, the existence of the divergence due to the scalars simplifies considerably the problem [34].

We are to compute the partition function,

$$\mathrm{Tr}\, P e^{-\beta H}, \tag{B.12}$$

on the Hilbert space constructed out of the vacuum with the 8 fermionic oscillators $a_I^{A\,\dagger}$, the gauge exponential wavefunctions $e^{ik_i \cdot c_i}$ and the 6 scalar zero modes $e^{ip_{IJ} \cdot \phi_{IJ}}$. The states $|k_i, a_I^A, p_{IJ}\rangle, k_i \in a_i \Lambda_W, p_{IJ} \in R^r$ provide a suitable Fock basis of eigenvalues of the Hamiltonian with energy $(g^2/4V)(\mathbf{k}^2 + \sum_{IJ} p_{IJ}^2)$. $P = (1/|W|) \sum_{g \epsilon W} g$ is the projector over Weyl invariant states. The Weyl group is generated by the reflections about simple roots

$$k \to k_i - (e_n \cdot k_i) e_n. \tag{B.13}$$

It takes the state $|k_i\rangle = e^{ik_i \cdot c_i}$ into $|k_i^g\rangle$, another state in the Fock space. It leaves invariant the states with $e_n \cdot k_i = 0$, i.e. the walls of the Weyl chambers. The partition function reads

$$\mathrm{Tr}\, P e^{-\beta H} = \frac{1}{|W|} \sum_{g \in W} \sum_{\mathbf{k}, a_I^A} e^{-(\beta g^2/4V)\mathbf{k}^2} \prod_{IJ} \int dp_{IJ} \langle k_i, a_I^A, p_{IJ} | g | k_i, a_I^A, p_{IJ} \rangle e^{-(\beta g^2/4V) p_{IJ}^2}. \tag{B.14}$$

Now, the matrix element $\langle k_i|g|k_i\rangle$ is different from zero only if k_i is left invariant by g; so we can restrict the sum over g to the little group $W_{\mathbf{k}}$ of $\mathbf{k}$ (the set of elements of W which leaves $\mathbf{k}$ fixed):

$$\frac{1}{|W|}\sum_{k_i\in a_i\Lambda_W} e^{-(\beta g^2/4V)\mathbf{k}^2}\sum_{g\in W_{\mathbf{k}}}\sum_{a_I^A}\prod_{IJ}\int dp_{IJ}\langle a_I^A,p_{IJ}|g|a_I^A,p_{IJ}\rangle e^{-(\beta g^2/4V)p_{IJ}^2}. \tag{B.15}$$

Since W acts freely in the interior of the Weyl chamber, $W_{\mathbf{k}}$ is trivial for such elements, becoming larger and larger on the various walls, walls of walls etc., and becomes the complete Weyl group only at $\mathbf{k}=\mathbf{0}$.

Let us compute the matrix element $\langle a_I^A,p_{IJ}|g|a_I^A,p_{IJ}\rangle$. On the fermionic Fock space, g acts as a linear transformation, thus giving a factor $\det(1+g)$ for each spinorial degree of freedom. The scalars contribution is

$$\langle p_{IJ}|g|p_{IJ}\rangle=\prod_{IJ}\int dp_{IJ}\int_0^R dx_{IJ}e^{-(\beta g^2/4V)[p_{IJ}^2+ip_{IJ}(x_{IJ}-x_{IJ}^g)]}, \tag{B.16}$$

where we have put a cutoff R to regularize the configuration space divergence of the scalars; whenever possible we will send $R\to\infty$. For each scalar degree of freedom, we find

$$\int_0^R dxe^{-\frac{1}{4\beta}x(1-g)^2x}=\frac{R^{n_g}}{\det'(1-g)}; \tag{B.17}$$

n_g is the number of eigenvalues of g equal to 1, and the prime means the exclusion of the zero eigenvalues.

To sum up, the total partition function reads

$$\sum_{k\in a_i\Lambda_W} e^{-(\beta g^2/4V)\mathbf{k}^2}\frac{1}{|W|}\sum_{g\epsilon W_{\mathbf{k}}} R^{n_g}\frac{\det^8(1+g)}{\det'^6(1-g)}. \tag{B.18}$$

Now, it is clearly difficult to evaluate exactly the coefficients. However, we need to take in this sum only the most divergent contribution in R, since the subleading terms disappear when we normalize to get physical quantities. For each $W_{\mathbf{k}}$ there is only one element with the maximum power n_g (and so with the maximum divergence): the identity. All the other elements contribute subleading terms. We see that the most divergent coefficient is always the same and henceforth the exact theta function,

$$\sum_{k_i\in a_i\Lambda_W} e^{-\sum_i(\beta g^2/4V)k_i^2}, \tag{B.19}$$

gets reconstructed, up to a multiplicative factor.

The full partition function splits into flux sector according to the projector (B.2); one can check that this restricts the sum over weight vectors exactly according to the definition given in Section 3. This argument shows that the $\theta=\hat{\mathbf{m}}=0$, $g\ll 1$ approximation of the free energy completely agrees with the results of this paper.

Finally, let us remark that by using factorization, 't Hooft's duality, and the Witten phenomenon (see Section 5), one can derive the result (4.60) from the $\theta=\hat{\mathbf{m}}=0$ free energy.

References

[1] C. Montonen and D. Olive, Phys. Lett. 72B (1977) 117.

[2] E. Witten and D. Olive, Phys. Lett. 78B (1978) 97.

[3] H. Osborn, Phys. Lett. 83B (1979) 321.

[4] F. Englert and P. Windey, Phys. Rev. D14 (1976) 2728; P. Goddard, J. Nuyts and D. Olive, Nucl. Phys. 125B (1977) 1.

[5] P. Rossi, Phys. Lett. Phys. Lett. 99B (1981) 229.

[6] J. Cardy and E. Rabinovici, Nucl. Phys. 205B (1982) 1; J. Cardy, Nucl. Phys. 205B (1982) 17.

[7] A. Sen, Phys. Lett. B329 (1994) 217.

[8] N. Seiberg and E. Witten, Nucl. Phys. B426 (1994) 19, ERRATUM: ibid. B430 (1994) 396; Nucl. Phys. B431 (1994) 484.

[9] K. Intriligator and N. Seiberg, preprint RU-94-68, IASSNS-HEP-94/67, hep-th/9408155; N. Seiberg, preprint RU-94-82, IASSNS-HEP-94/98, hep-th/9411149.

[10] A. Font, L. Ibanez, D. Lust and F. Quevedo, Phys. Lett. 249B (1990) 35; S.J. Rey, Phys. Rev. D43 (1991) 526.

[11] A. Strominger, Nucl. Phys. 343B (1990) 167; C. Callan, J. Harvey and A. Strominger, Nucl. Phys. 359B (1991) 40; M. Duff, Class. Quant. Grav. 5 (1988) 189; M. Duff and J. Lu, Nucl. Phys. 354B (1991) 129; 357B (1991) 354; Phys. Rev. Lett. 66 (1991) 1402; Class. Quant. Grav. 9 (1991) 1; M. Duff, R. Khuri and J. Lu, Nucl. Phys. 377B (1992) 281; J. Dixon, M. Duff and J. Plefka, preprint CTP-TAMU-60/92, hep-th/9208055; I. Pesando and A. Tollsten, Phys. Lett. 274B (1992) 374.

[12] A. Sen, Int. J. Mod. Phys. A9 (1994) 3707; J.H. Schwarz and A. Sen, Nucl. Phys. 404B (1993) 109.

[13] J. Harvey, J. Gauntlett and J. Liu, Nucl. Phys. 409B (1993) 363; R. Khuri, Phys. Lett. 259B (1991) 261; J. Gauntlett and J. Harvey, hep-th/9407111.

[14] A. Giveon, M. Porrati and E. Rabinovici, Phys. Rep. 244 (1994) 77.

[15] I. Bakas, preprint CERN-TH.7144/94, hep-th/9402016; A. Sen, preprint TIFR/TH-94-19, hep-th/9408083; J.H. Schwarz, preprint CALT-68-1965, hep-th/9411178.

[16] C. Vafa and E. Witten, Nucl. Phys. B431 (1994) 3.

[17] L. Girardello, A. Giveon, M. Porrati and A. Zaffaroni, Phys. Lett. 334B (1994) 331.

[18] G. 't Hooft, Nucl. Phys. 153B (1979) 141; *Lectures given at the 1980 Schladming Winter School in Physics*, Acta Phys. Austr. Suppl. XXII (1980) 531 (Springer-Verlag, Berlin, 1980).

[19] E. Witten, Phys. Lett. 86B (1979) 283.

[20] L. Brink, J.H. Schwarz and J. Scherk, Nucl. Phys. 121B (1977) 77.

[21] M. Grisaru, M. Roček and W. Siegel, Phys. Rev. Lett. 45 (1980) 1063; L.V. Avdeev, D.V. Tarasov and A.A. Vladimirov, Phys. Lett. 96B (1980) 94; W.E. Caswell and D. Zanon, Phys. Lett. 100B (1981) 152.

[22] M.A. Shifman, A.I. Vainshtein and M.I. Vysotsky, Nucl. Phys. 254B (1985) 619; N. Seiberg, Phys. Lett. 206B (1988) 75.

[23] E. Witten, Nucl. Phys. 202B (1982) 253.

[24] K.G. Wilson, Phys. Rev. B4 (1971) 3174; K.G. Wilson and J. Kogut, Phys. Rep. 12 (1974) 75.

[25] J. Polchinski, Nucl. Phys. 231B (1984) 269.

[26] G. Gallavotti, Rev. Mod. Phys. 57 (1985) 471.

[27] C. Becchi, in *Elementary Particles, Quantum Field Theory and Statistical Mechanics*, ed. M. Bonini, G. Marchesini and E. Onofri, Parma University, 1993.

[28] P.L. White, Class. Quant. Grav. 9 (1992) 413.

[29] T. Appelquist and J. Carrazzone, Phys. Rev. D11 (1975) 2856.

[30] L. Girardello and A. Zaffaroni, Nucl. Phys. 424B (1994) 219.

[31] A. Klemm, W. Lerche, S. Yankielowicz and S. Theisen, preprint CERN-TH.7495/94, LMU-TPW 94/16, hep-th/9411048; preprint CERN-Th.7538/94, LMU-TPW 94/22, hep-th/9412158; P.C. Argyres and A.E. Faraggi, preprint IASSNS-HEP-94/94, hep-th/9411057.

[32] M. Flato and C. Fronsdal, Lett. Math. Phys. 8 (1984) 159; V.K. Dobrev and V.B. Petkova, Phys. Lett. 162B (1985) 127.

[33] J.A. Harvey, G. Moore and A. Strominger, hep-th/9501022; M. Bershadsky, A. Johansen, V. Sadov and C. Vafa, preprint HUTP-95/A004.

[34] C. Imbimbo and S. Mukhi, private communication.

[35] M. Luscher, Nucl. Phys. 219B (1983) 261.